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KNOTS ON ONCE-PUNCTURED TORUS FIBERS

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KNOTS ON ONCE-PUNCTURED TORUS FIBERS

by

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Dedicated to my grandparents:

Milo, Dorothy, Terry, and Marie.

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To all the many friends and people who are a part of or have passed through my life these years in Austin and Princeton, including my fellow graduate students and especially all my officemates, I raise a glass. Many of you have been there for me in many ways at many times, and I can only hope to do the same for all y'all. Cheers.

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KNOTS ON ONCE-PUNCTURED TORUS FIBERS

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We study knots that lie as essential simple closed curves on the fiber of a genus one fibered knot in S^3. We determine certain surgery descriptions of these knots that enable estimates on volumes of these knots. We also develop an algorithm to list all closed essential surfaces in the complement of a given knot in this family. Relationships between the volumes of such knots and the surfaces in their exteriors is then examined.

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Chapter 1

Introduction

As shown by Lickorish [14] and Wallace [21], any closed orientable 3-manifold may be obtained by Dehn surgery on some link in S^3. It is however not the case that every such 3-manifold may be obtained by Dehn surgery on a knot. One is prompted to then ask which 3-manifolds may be obtained by Dehn surgery on a knot. Similarly one may ask to classify the knots with some non-trivial Dehn surgery that yields a certain 3-manifold or a 3-manifold with a certain property.

Perhaps the first two main questions along these lines are whether non-trivial surgery on a knot in S^3 can return S^3 and more generally whether non-trivial surgery on a knot in S^3 can yield a simply connected manifold. These are both very deep and difficult questions. The former was answered in the negative by Gordon and Luecke [8]. The latter (known as Property P) was more recently answered in the negative by Kronheimer, Mrowka, Ozsváth, and Szabó [13].

The next "simplest" class of 3-manifolds are the lens spaces. They are simple in the sense that they nearly simply connected having cyclic fundamental group and they have Heegaard genus 1. There are many examples of knots with non-trivial surgeries yielding lens spaces. Berge [3] characterizes all known examples of such knots and lists the knots with this characterization. Conjecturally this list is complete. Though there has been much interest recently in knots with lens space surgeries, this conjecture remains wide open.

We examine a certain family of these knots, the knots that lie as essential simple closed curves on the fiber of the Left Handed Trefoil, Right Handed Trefoil, or Figure Eight Knot, and determine a marked difference between this family of knots and the others that Berge describes. After reviewing definitions and preliminary material in Chapter 2, we show in Chapter 3 that the volumes of the hyperbolic knots in this family is unbounded. In Chapter 4 we develop a practical algorithm to list all the closed essential surfaces in the complements of these knots. In Chapter 5 we tie the previous two chapters together by finding relationships between volumes of the knots and the genera of the closed essential surface (if any) contained in their complements. We supplement this with two appendices. The first gives descriptions of the other known knots with lens space surgeries as arising from surgeries on the Minimally Twisted 5 Chain link. The second gives descriptions of the knots of our primary interest as the closures of positive (or negative) braids.

Chapter 2

Definitions

Here we collect some of the basic definitions and concepts that we will use throughout this paper. See any of the many knot theory and low dimensional topology texts such as Rolfsen [17], Burde and Zieschang [5], or Kawauchi [11] for a more thorough treatment of surgery, lens spaces, and fiber bundles. See Thurston [20] or Benedetti and Petronio [2] for an introductions to hyperbolic 3-manifolds.

2.1 Surgery on Knots and Links

Let M be a compact, orientable 3-manifold with a toroidal boundary component $\partial_0 M$.

Definition 2.1.1. A *slope* on $\partial_0 M$ is an isotopy class of unoriented simple closed curves.

Due to their lack of orientation, slopes are in a 2 to 1 correspondence with primitive elements of $\pi_1(\partial_0 M) \cong H_1(\partial_0 M; \mathbb{Z}) \cong \mathbb{Z}^2$. Given a choice of basis for $H_1(\partial_0 M)$, say $\{\mu, \lambda\}$, we may identify the set of slopes on $\partial_0 M$ with the extended rationals $\mathbb{Q} \cup \infty$: Let c be an oriented simple closed curve on

$\partial_0 M$ representing the slope ρ. If $[c] = p\mu + q\lambda \in H_1(\partial_0 M)$, then we identify ρ with $\frac{p}{q} \in \mathbb{Q} \cup \infty$.

Distinctions between isotopy classes, homology classes, and representative simple closed curves are often blurred as context will make clear any ambiguities. When a basis for $H_1(\partial_0 M)$ is understood, an extended rational number will be used synonymously with its corresponding slope.

Definition 2.1.2. Let ρ be a slope on $\partial_0 M$ of M. Then we may construct the 3-manifold $M(\rho)$, the ρ-*Dehn filling* of M on $\partial_0 M$, by attaching a solid torus to $\partial_0 M$ as follows. Let $h : \partial(S^1 \times D^2) \to \partial_0 M$ be a homeomorphism such that $h(* \times \partial D^2)$ represents ρ. Then $M(\rho) \cong M \cup_h (S^1 \times D^2)$.

Notice that $M(\rho)$ depends up to homeomorphism only on ρ. If ρ is identified with $\frac{p}{q}$, then we may write $M(\frac{p}{q})$ instead of $M(\rho)$.

Definition 2.1.3. Assume M has k toroidal boundary components, $\partial_i M$ for $i = 1, \ldots, k$, with slopes ρ_i on $\partial_i M$ for $i = 1, \ldots, k$ then we may form the *slope vector* $\overline{\rho} = (\rho_1, \ldots, \rho_k)$. Then $M(\overline{\rho})$ is the $\overline{\rho}$-*Dehn filling* of M where

$$M(\overline{\rho}) \cong M(\rho_1)(\rho_2)\ldots(\rho_k),$$

the successive Dehn fillings of the boundary components of M.

Remark 2.1.1. The Dehn fillings of $\overline{\rho}$-Dehn filling may be done in any order or all at once.

Definition 2.1.4. Let L be the ambient isotopy class of embeddings

$$f\colon \bigcup_{i=1}^{k} S_i^1 \to M$$

of k circles so that $f(S_i^1)$ has a regular neighborhood for each i. Then L is a k *component (tame) link* in M. If L has only one component, then we say L is a *knot* in M.

Definition 2.1.5. Let L be a link in S^3. Let $N(L)$ be a regular open neighborhood of L. Then the *exterior of* L is the manifold $X_L = S^3 - N(L)$.

Remark 2.1.2. If L is a k component link, then ∂X_L is a disjoint union of k tori.

Definition 2.1.6. Let L be a k component link in S^3. Let $\partial X_L = \bigcup_{i=1}^k \partial_i X_L$, and let $\overline{\rho}$ be a slope vector of slopes ρ_i on $\partial_i X_L$. We say the manifold $X_L(\overline{\rho})$ is obtained by *$\overline{\rho}$-Dehn surgery on L.*

Let L_i be the ith component of a link L in S^3 with some chosen orientation. Given a non-trivial simple closed curve m_i on $\partial N(L_i)$ that bounds a disk in $\overline{N(L_i)}$, orient m_i so that the linking number of m_i and L_i is $+1$. For a simple closed curve l_i on $\partial N(L_i)$ parallel to L_i in $\overline{N(L_i)}$, orient l_i coherently with L_i. Note that these orientations (with the orientation on $\partial N(L_i)$ induced by the orientation on $N(L_i)$) imply the algebraic intersection number $m_i \cdot l_i = +1$. See Figure 2.1 for a depiction of L_i, m_i, and l_i.

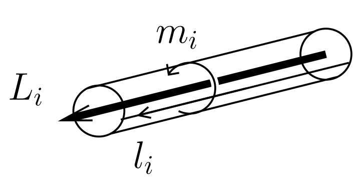


Figure 2.1: A meridian and longitude for L_i.

Definition 2.1.7. We say that m_i is a *meridian* of L_i and that l_i is a *longitude*. If l_i is the boundary of an orientable surface in $X(L_i)$ (a Seifert surface), then we say the pair $\{m_i, l_i\}$ is a *standard meridian-longitude* pair. We use these terms for the corresponding unoriented curves, elements of $H_1(\partial N(L_i))$, and slopes as well. For $\mu_i = [m_i], \lambda_i = [l_i] \in H_1(\partial N(L_i))$, we say $\{\mu_i, \lambda_i\}$ is the *standard basis* for $H_1(\partial N(L_i))$.

Remark 2.1.3. Notice that choosing the opposite orientation of L_i changes the orientation of the meridian and hence the longitude too. This change however has no net effect on a slope $\frac{p}{q}$.

More generally:

Definition 2.1.8. A pair of (oriented) simple closed curves m and l on a torus T such that $m \cdot l = +1$ form a *basis* for T, and their homology classes $[m]$ and $[l] \in H_1(T)$ form a basis for $H_1(T)$.

2.1.1 Dehn Twists

Let c be a simple closed curve on the orientable surface S.

Definition 2.1.9. The homeomorphism $S \to S$ defined by cutting S along c and regluing along c after rotating a full counter-clockwise turn is called the *left-handed Dehn twist* of S about c.

See Figure 2.2 for a sketch of the effect of this homeomorphism. Note that this homeomorphism is isotopic to the identity outside a neighborhood of c.

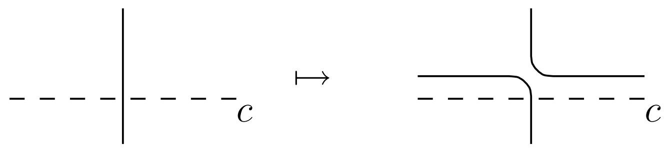


Figure 2.2: Left-handed Dehn twist about c.

Let R be an annulus embedded in an orientable 3-manifold M and set $L_{+1} \cup L_{-1} = \partial R$. Let m_i and l_i be the the standard meridian on $\partial N(L_i)$ and longitude on $\partial N(L_i)$ induced by R for $i = \pm 1$.

In these bases, Dehn surgery of $\frac{1}{n}$ on L_{+1} and $-\frac{1}{n}$ on L_{-1} yields the homeomorphism $M \cong (M - N(L_{+1} \cup L_{-1}))(\frac{1}{n}, -\frac{1}{n})$. This homeomorphism is realized by cutting $M - N(L_{+1} \cup L_{-1})$ along R, regluing along R after spinning n times counter-clockwise, and filling the two toroidal boundary components incident to R trivially (see Figure 2.3 for the case $n = 2$). Note that this homeomorphism is isotopic to the identity outside a neighborhood of R.

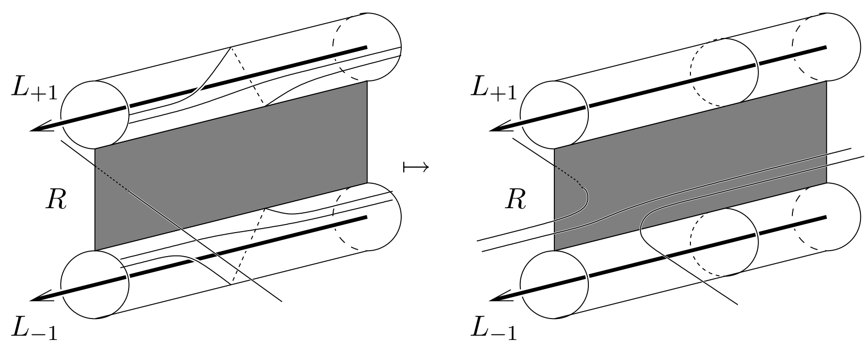


Figure 2.3: Twisting along an annulus R by surgery.

Remark 2.1.4. If a surface S intersects the annulus R along a simple closed curve c which is essential in R, then the homeomorphism

$$M \to (M - N(L_{+1} \cup L_{-1}))(\frac{1}{n}, -\frac{1}{n})$$

restricted to S is the composition of n left-handed Dehn twists.

2.1.2 Tangles

Definition 2.1.10. A *tangle* (B, t) is a pair consisting of a punctured 3-sphere B and a properly embedded collection t of disjoint arcs and simple closed curves. Two tangles (B_1, t_1) and (B_2, t_2) are homeomorphic if there is a homeomorphism of pairs

$$h\colon (B_1, t_1) \to (B_2, t_2).$$

A boundary component $(\partial B_0, t \cap \partial B_0)$ of a tangle (B, t) is a sphere ∂B_0 together with some finite number of points $t \cap \partial B_0$.

Definition 2.1.11. Given a sphere S and set p of four distinct points on S, a *framing* of (S, p) is an ordered pair of (unoriented) simple closed curves $(\hat{m}, \hat{l})$ on $S - N(p)$ such that each curve separates different pairs of points of p.

Let (S, p) be a sphere with four points with framing $(\hat{m}, \hat{l})$. The double cover of S branched over p is a torus. Single components, say m and l, of the lifts of the framing curves $\hat{m}$ and $\hat{l}$ when oriented so that $m \cdot l = +1$ (with respect to the orientation of the torus) form a basis for the torus. Similarly, a basis on a torus induces a framing on the sphere with four points obtained by the quotient of an involution that fixes four points on the torus. See Figure 2.4.

2.2 Surfaces and Bundles

The notation and terminology we set up in §§2.2.1 and 2.2.2 is borrowed from Culler, Jaco, and Rubinstein [6] so that our work will be consistent with theirs.

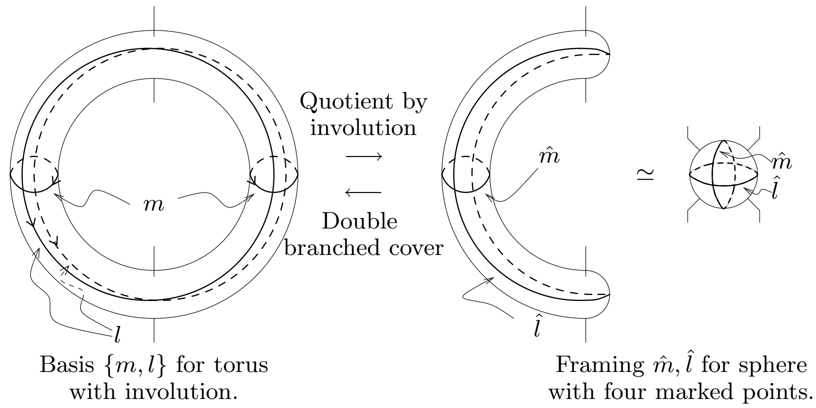


Figure 2.4: The correspondence of bases and framings.

2.2.1 The Once-Punctured Torus and $\mathrm{SL}_2(\mathbb{Z})$.

Let T be the once-punctured torus shown (with the appropriate identifications of edges) in Figure 2.5. Also shown are the oriented simple closed curves a and b and arcs a_+, a_-, b_+, and b_-.

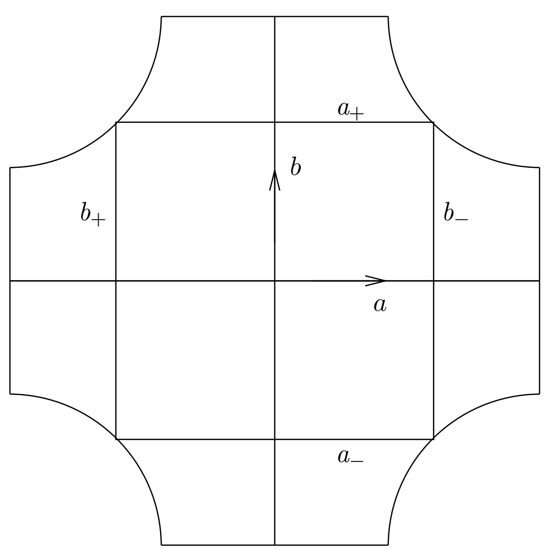


Figure 2.5: The once-punctured torus T.

Let α and β be the left-handed Dehn twists along a and b respectively.

Then α and β generate the group $\mathcal{H}^+(T)$ of orientation preserving homeomorphisms up to isotopy. This group is naturally identified with the orientation preserving homomorphisms of $H_1(T) \cong \mathbb{Z}^2$, $\mathrm{SL}_2(\mathbb{Z})$. Assuming $H_1(T)$ is generated by $[a]$ and $[b]$, this identification is done by

$$\alpha \mapsto A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \text{ and}$$
$$\beta \mapsto B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Consequentially, these two matrices generate $\mathrm{SL}_2(\mathbb{Z})$.

$\mathrm{SL}_2(\mathbb{Z})$ is also generated by the two matrices

$$P = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \text{ and } Q = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}.$$

This gives the presentation

$$\mathrm{SL}_2(\mathbb{Z}) = \langle P, Q | P^4 = Q^6 = I, P^2Q^3 = I \rangle.$$

Note that P is identified with the isotopy class of the homeomorphism ϕ of T that is induced by rotating Figure 2.5 by $\frac{\pi}{2}$. Also note that $BAB = P$ and $P^2 = -I$.

2.2.2 Once-Punctured Torus Bundles

If η is an orientation preserving homeomorphism of T, then we may form the once-punctured torus bundle

$$M = T \times I/_\eta$$

defined by identifying $T \times \{1\}$ with $T \times \{0\}$ of $T \times I$ by $(x, 1) \mapsto (\eta(x), 0)$.

If we express η as a composition of such homeomorphisms $\eta = \eta_n\eta_{n-1}\dots\eta_2\eta_1$, then M may be divided into "blocks" as

$$M = T \times I/_{\eta_1} T \times I/_{\eta_2} \dots /_{\eta_{n-1}} T \times I/_{\eta_n}$$

where $T \times \{1\}$ of the ith block is identified with $T \times \{0\}$ of the $i+1$th (mod n) block according to $(x,1)_i \mapsto (\eta_i(x), 0)_{i+1}$.

If H_i is identified with the isotopy class of the homeomorphism η_i, then M has characteristic class $[H_nH_{n-1}\dots H_2H_1]$.

2.2.3 Essential Curves and Surfaces

Definition 2.2.1. Let c be a simple closed curve on a compact orientable surface S. If c does not bound a disk in S and does not bound an annulus with a component of ∂S, then c is an *essential curve*.

Definition 2.2.2. Let S be a properly embedded compact orientable surface in a compact orientable 3-manifold. If S is incompressible, ∂-incompressible, and not ∂-parallel, then S is an *essential surface*.

Definition 2.2.3. If M is a compact orientable 3-manifold that contains no essential closed orientable surfaces, then M is *small*.

2.3 Continued Fractions

Definition 2.3.1. A *continued fraction expansion* $[\overline{b}] = [b_1, b_2, \ldots, b_k]$ of $\frac{p}{q}$ is defined as follows:

$$\frac{p}{q} = [b_1, b_2, \ldots, b_k] = \cfrac{1}{b_1 - \cfrac{1}{b_2 - \cfrac{1}{\cdots - \cfrac{1}{b_k}}}}$$

where $b_i \in \mathbb{Z}$. (We will formally make sense of the case that $b_k = 0$.) The b_i terms are referred to as the *coefficients* of the continued fraction expansion. We say the continued fraction expansion $[\overline{b}]$ has *length* k.

Remark 2.3.1. There are various definitions for continued fractions. This choice of definition is useful for our purposes.

A rational number has many continued fraction expansions. We use the notation $\overline{b}$ to indicate a specific sequence of coefficients and $[\overline{b}]$ to indicate the corresponding rational number together with the specific sequence of coefficients.

Definition 2.3.2. A given continued fraction expansion $[\overline{b}] = [b_1, b_2, \ldots, b_k]$ for $\frac{p}{q}$ has *partial fractions* $\frac{p_i}{q_i} = [b_1, b_2, \ldots, b_i]$ for $i = 1, \ldots, k$. We set the partial fraction $\frac{p_0}{q_0} = [\emptyset] = \frac{0}{1}$.

Definition 2.3.3. Say $[b_1, b_2, \ldots, b_k]$ is a *minimal continued fraction expansion (MCFE)* for $\frac{p}{q}$ if $|b_i| \geq 2$ for all $i = 1, \ldots, k$.

Note that in order for $\frac{p}{q}$ to have a MCFE, it must be that $|\frac{p}{q}| < 1$.

Definition 2.3.4. Say $[b_1, b_2, \ldots, b_k]$ is the *simple continued fraction expansion (SCFE)* for $\frac{p}{q}$ if the coefficients alternate signs, $|b_k| \geq 2$, and $b_i \neq 0$ for $i \geq 2$.

Notice that though continued fraction expansions are not unique, each rational number $\frac{p}{q}$ does have a unique SCFE. Furthermore, the SCFE for $\frac{p}{q}$ has 0 as its first coefficient if and only if $|\frac{p}{q}| \geq 1$ (see Lemma 2.3.1).

Definition 2.3.5. Let $\mathcal{D}$ be the diagram shown in Figure 2.6. $\mathcal{D}$ is a disk with the extended rational numbers marked on its boundary. An edge joins vertices $\frac{a}{b}$ to $\frac{c}{d}$ if and only if $ad - bc = \pm 1$. The edge from $\frac{a}{b}$ to $\frac{c}{d}$ is the 'long' edge of the triangle whose third vertex is $\frac{a+c}{b+d}$.

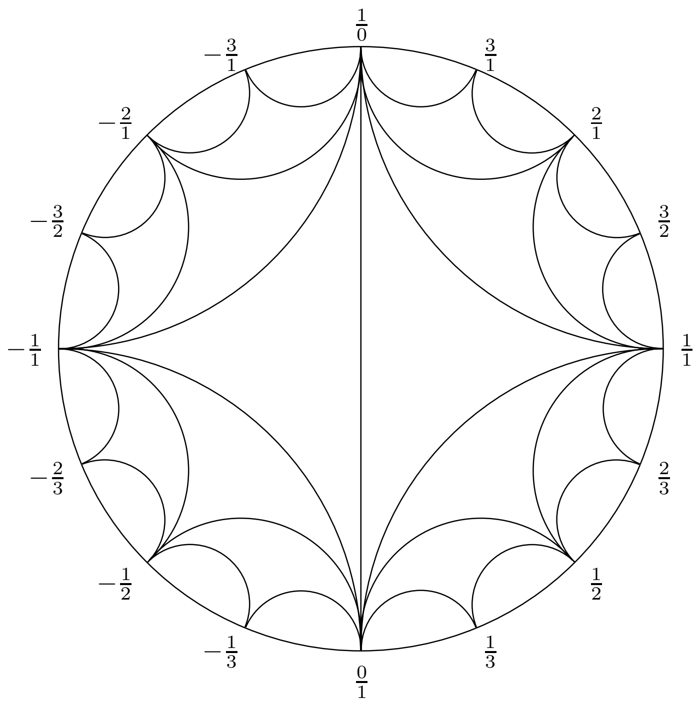


Figure 2.6: The diagram $\mathcal{D}$

Remark 2.3.2. $\mathcal{D}$ is in fact the classical diagram of the action of the modular group $\mathrm{PSL}_2(\mathbb{Z})$ on the hyperbolic plane $\mathbb{H}^2$ viewed as the Poincaré disk. We utilize the connections between $\mathcal{D}$, $\mathrm{SL}_2(\mathbb{Z})$ (and $\mathrm{PSL}_2(\mathbb{Z})$), and continued fraction expansions in the vein of Floyd-Hatcher [7] and Hatcher-Thurston [10] and borrow some of their terminology. One may look to Kirby-Melvin [12] for a more thorough treatment of these connections.

Definition 2.3.6. An *(oriented) edge-path* $E_{\overline{b}} = \{e_0, e_1, e_2, \ldots, e_k\}$ from $\frac{1}{0}$ to $\frac{p}{q}$ where e_0 is the edge from $\frac{1}{0}$ to $\frac{0}{1}$ corresponds uniquely to a continued fraction expansion $[b_1, b_2, \ldots, b_k]$ for $\frac{p}{q}$ where the the end points of each edge e_i are the partial fractions $\frac{p_{i-1}}{q_{i-1}}$ and $\frac{p_i}{q_i}$. At the vertex $\frac{p_{i-1}}{q_{i-1}}$, facing inward, the edge e_i is $|b_i|$ triangles apart from e_{i-1}: to the left if $b_i < 0$ and to the right if $b_i > 0$. If $b_i = 0$ the edge e_i retraces the edge e_{i-1}.

See Figure 2.7 for an example of the edge-path corresponding to $[2, -1]$.

Definition 2.3.7. An edge-path $E_{\overline{m}}$ associated to a MCFE $[\overline{m}]$ is a *minimal edge-path.*

Remark 2.3.3. A minimal edge-path $E_{\overline{m}}$ is minimal in the sense that no triangle of $\mathcal{D}$ has two consecutive edges of $E_{\overline{m}}$ on its boundary.

Minimal edge paths from $\frac{1}{0}$ to $\frac{p}{q}$ are contained in the finite subcomplex $E_{\frac{p}{q}}$ of $\mathcal{D}$ comprised of the edge-path associated to the SCFE $[a_1, a_2, \ldots, a_l]$ for $\frac{p}{q}$ and the triangles between its edges. See Figure 2.8. Observe that a minimal edge path may only involve the heavier edges of the larger triangles.

Remark 2.3.4. It follows that there are only finitely many MCFEs for any rational number $\frac{p}{q}$. Furthermore, as noted in [10], the number of minimal edge

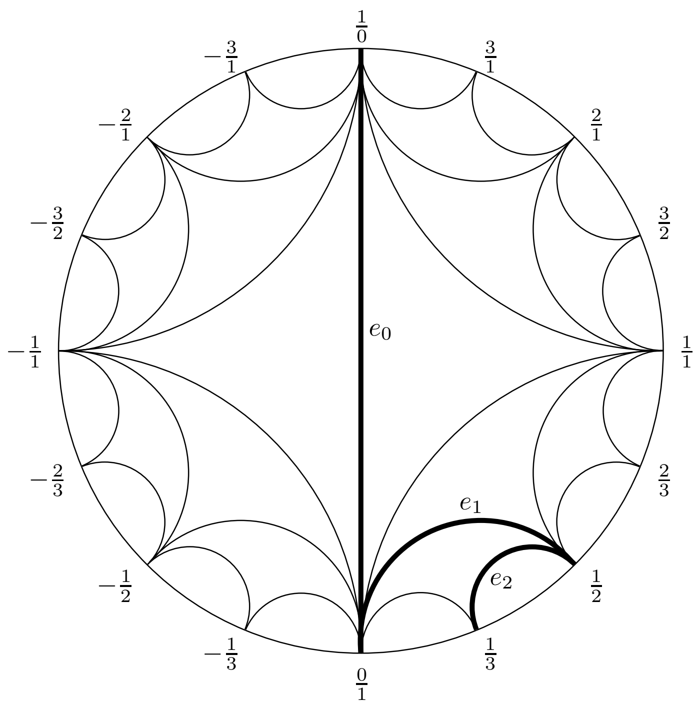


Figure 2.7: The edge-path $E_{[2,-1]}$.

paths from $\frac{1}{0}$ to $\frac{p}{q}$ (and hence the number of MCFEs for $\frac{p}{q}$) is the number ϕ_k defined recursively by:

$$\phi_i = \begin{cases} \phi_{i-2} + \phi_{i-1}, & |a_i| > 1 \\ \phi_{i-3} + \phi_{i-2}, & |a_i| = 1, \end{cases}$$
$$\phi_0 = \phi_{-1} = 1,$$
$$\phi_{-2} = 0.$$

Lemma 2.3.1. *Let $[\bar{b}] = [b_1, b_2, \ldots, b_k]$ be the SCFE for $\frac{p}{q} \neq \pm 1$. If $b_1 \neq 0$ then $|\frac{p}{q}| < 1$ and $\operatorname{sign}(\frac{p}{q}) = \operatorname{sign}(b_1)$. If $b_1 = 0$ then $|\frac{p}{q}| > 1$ and $\operatorname{sign}(\frac{p}{q}) = -\operatorname{sign}(b_2)$.*

Proof. This is immediate when one considers the edge-path in $\mathcal{D}$ associated to a SCFE. $\square$

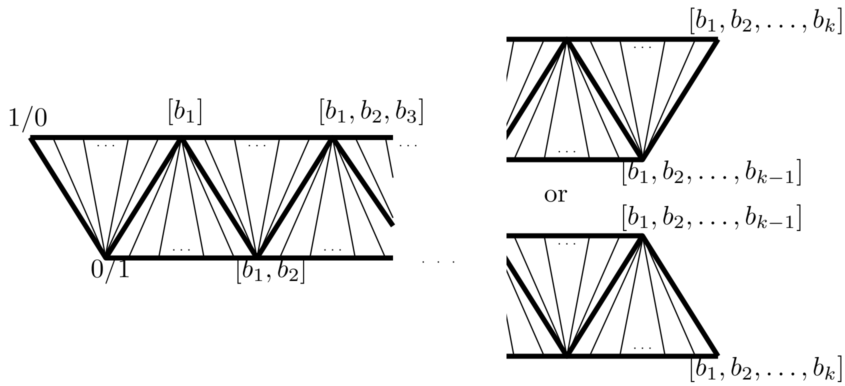


Figure 2.8: The subcomplex of $\mathcal{D}$ associated to a SCFE.

Lemma 2.3.2. *If the three partial fractions p_l/q_l, p_{l-1}/q_{l-1}, and p_{l-2}/q_{l-2} of $[a_1, \ldots, a_{l-2}, a_{l-1}, a_l] = p_l/q_l$ equal the three partial fractions p'_k/q'_k, p'_{k-1}/q'_{k-1}, and p'_{k-2}/q'_{k-2} of $[b_1, \ldots, b_{k-2}, b_{k-1}, b_k] = p'_k/q'_k$ respectively, then $a_l = b_k$.*

Proof. The three final partial sums determine the final two edges of the associated edge-paths. With these partial sums equal, the edges are the same, and hence they have the same number of triangles between them. This number is the final coefficient $a_l = b_k$. □

2.3.1 Relating Continued Fraction Expansions

Let $[\overline{a}] = [a_1, a_2, \ldots, a_l]$ and $[\overline{b}] = [b_1, b_2, \ldots, b_k]$ be two continued fraction expansions for the same rational number. Then $[\overline{a}]$ may be obtained from $[\overline{b}]$

by a finite sequence of the following *elementary moves* and their inverses:

$$[\ldots, r, s, \ldots] \mapsto [\ldots, r \pm 1, \pm 1, s \pm 1, \ldots], \tag{CF 1}$$

$$[\ldots, r] \mapsto [\ldots, r \pm 1, \pm 1], \tag{CF 1'}$$

$$[\ldots, r + s, \ldots] \mapsto [\ldots, r, 0, s, \ldots], \tag{CF 2}$$

$$[\ldots, r] \mapsto [\ldots, r, s, 0]. \tag{CF 2'}$$

These may be understood via the corresponding moves on edge-paths $E_{\overline{a}}$ and $E_{\overline{b}}$.

Lemma 2.3.3. *Move (CF 2') may be obtained from the other three moves.*

Proof. For move (CF 2'), if $s = 0$ then we may directly apply move (CF 2).

If $s = \mp 1$, then

$$\begin{aligned} [\ldots, r] &\mapsto [\ldots, r \pm 1, \pm 1] && \text{by (CF 1')} \\ &\mapsto [\ldots, r, 0, \pm 1, \pm 1] && \text{by (CF 2)} \\ &\mapsto [\ldots, r, \mp 1, 0] && \text{by (CF 1).} \end{aligned}$$

If $|s| > 1$, then we induct:

$$\begin{aligned} [\ldots, r, s \pm 1, 0] &\mapsto [\ldots, r, s \pm 1, 0, \mp 1, 0] && \text{by previous case} \\ &\mapsto [\ldots, r, s, 0] && \text{by (CF 2).} \end{aligned}$$

□

Let $[\overline{a}] = [a_1, a_2, \ldots, a_k]$ be the SCFE for $\frac{p}{q}$. Then any MCFE for $\frac{p}{q}$ may be obtained from $[\overline{a}]$ by a sequence of moves on non-adjacent coefficients $a_i \neq 0$ of $[\overline{a}]$ of the following forms

$$[\ldots, a_{i-1}, a_i, a_{i+1}, \ldots] \mapsto [\ldots, a_{i-1} \pm 1, \underbrace{\pm 2, \pm 2, \ldots, \pm 2}_{|a_i - 1|}, a_{i+1} \pm 1, \ldots], \tag{M}$$

for $i \neq 1$ or k and

$$[\ldots, a_{k-1}, a_k] \mapsto [\ldots, a_{k-1} \pm 1, \underbrace{\pm 2, \pm 2, \ldots, \pm 2}_{|a_k - 1|}]. \tag{M'}$$

When (M) or (M') is applied to a coefficient $a_i = \pm 1$, it is tantamount to applying the inverses of the move (CF 1) or (CF 1') respectively. We use the "+" if $a_i < 0$ and the "−" if $a_i > 0$. The moves (M) and (M') may be obtained as repeated applications of move (CF 1) together with one application of move (CF 1') for (M'). Again, these may be understood via the corresponding edge-paths.

Lemma 2.3.4. *Assume $[\overline{b}] = [b_1, \ldots, b_k]$ is a SCFE for $\frac{p}{q}$ and $[\overline{a}] = [a_1, \ldots, a_l]$ is a MCFE for $\frac{p}{q}$. If $\frac{r}{s} = [\overline{b}, N] = [\overline{a}, N']$ for some integers N and N', then either $N' = N$ or $N' = N - \operatorname{sign}(b_k)$.*

Proof. Consider the edge-paths in $\mathcal{D}$ corresponding to these two continued fraction expansions of $\frac{r}{s}$. Since both of these continued fractions have the same penultimate partial sum of $\frac{p}{q}$, their edge-paths $E_{\overline{b},N}$ and $E_{\overline{a},N'}$ share the same final edge. The integers N and N' describe how far apart are the final edges of $E_{\overline{b},N}$ and $E_{\overline{a},N'}$ from the final edges of $E_{\overline{b}}$ and $E_{\overline{a}}$.

Recall that any MCFE for a rational number is obtained from its SCFE by a sequence of moves (M) and (M'). If move (M') is not used in obtaining

$[\overline{a}]$ from $[\overline{b}]$ then $E_{\overline{a}}$ and $E_{\overline{b}}$ have the same final edge. In this case, $N' = N$. This case may also be viewed as a consequence of Lemma 2.3.2.

If move (M') is used in obtaining $[\overline{a}]$ from $[\overline{b}]$ then the final edges of $E_{\overline{a}}$ and $E_{\overline{b}}$ are distinct edges of a triangle Δ in $\mathcal{D}$ sharing the vertex at $\frac{p}{q}$. The final edge of $E_{\overline{b}}$ is b_k triangles (counted with sign) apart from the penultimate edge of $E_{\overline{b}}$. These b_k triangles cross third edge of Δ, and hence the final edge of $E_{\overline{b}}$ is $\text{sign}(b_k)$ triangles apart from the third edge of Δ. Therefore the final edge of $E_{\overline{b}}$ is $-\text{sign}(b_k)$ triangles apart from the final edge of $E_{\overline{a}}$. This implies that the final edge of $E_{\overline{b},N}$ is $N + -\text{sign}(b_k)$ edges apart from the final edge of $E_{\overline{a}}$. Thus the final edge of $E_{\overline{a},N'}$ is $N - \text{sign}(b_k)$ edges apart from the final edge of $E_{\overline{a}}$ so that $N' = N - \text{sign}(b_k)$. $\square$

Coefficient Sums.

Definition 2.3.8. If $[\overline{b}] = [b_1, b_2, \ldots, b_k]$ is a continued fraction expansion, then $\sigma(\overline{b}) = \sum_{i=1}^{k} b_i$ is the *coefficient sum* of $[\overline{b}]$.

Assume $[\overline{b'}]$ is obtained from $[\overline{b}]$ by an elementary move. Then the coefficient sums differ as follows:

$$\sigma(\overline{b'}) - \sigma(\overline{b}) = \begin{cases} \pm 3 \text{ if } [\overline{b'}] \text{ is obtained by (CF 1)}, \\ \pm 2 \text{ if } [\overline{b'}] \text{ is obtained by (CF 1')}, \\ 0 \text{ if } [\overline{b'}] \text{ is obtained by (CF 2)}. \end{cases}$$

The difference varies, however, if $[\overline{b'}]$ is obtained from $[\overline{b}]$ by move (CF 2'). Also if $[\overline{b'}]$ is obtained from $[\overline{b}]$ by move (M) at b_i for $1 < i < k$ or move (M') at b_k, then

$$\sigma(\overline{b'}) - \sigma(\overline{b}) = \begin{cases} -3b_i & \text{if } i \neq k, \\ -3b_k + 1 & \text{if } b_k > 0, \\ -3b_k - 1 & \text{if } b_k < 0. \end{cases}$$

Lengths.

Assume $[\overline{b'}]$ is obtained from $[\overline{b}]$ by an elementary move. Then the lengths differ as follows:

$$\text{length}(\overline{b'}) - \text{length}(\overline{b}) = \begin{cases} 1 \text{ if } [\overline{b'}] \text{ is obtained by (CF 1) or (CF 1')}, \\ 2 \text{ if } [\overline{b'}] \text{ is obtained by (CF 2) or (CF 2')}. \end{cases}$$

Also if $[\overline{b'}]$ is obtained from $[\overline{b}]$ by move (M) or (M') at b_i then

$$\text{length}(\overline{b'}) - \text{length}(\overline{b}) = |b_i| - 2.$$

2.3.2 $\mathrm{SL}_2(\mathbb{Z})$ and Continued Fraction Expansions

Lemma 2.3.5. *Let* $W = \begin{pmatrix} x & t \\ y & u \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. *Then*

1. *depending on the parity of* k

$$\begin{array}{ccc} W = \pm B^{n_1} A^{n_2} \dots B^{n_k} & \textit{or} & \pm BABA^{n_1} B^{n_2} \dots B^{n_k} \\ & \textit{if and only if} & \\ \frac{x}{y} = [n_1, n_2, \dots, n_k] & \textit{and} & \frac{t}{u} = [n_1, n_2, \dots, n_{k-1}], \end{array}$$

   and

2. *depending on the parity of* k

$$\begin{array}{ccc} W = \pm B^{n_1} A^{n_2} \dots A^{n_k} & \textit{or} & \pm BABA^{n_1} B^{n_2} \dots A^{n_k} \\ & \textit{if and only if} & \\ \frac{x}{y} = [n_1, n_2, \dots, n_{k-1}] & \textit{and} & \frac{t}{u} = [n_1, n_2, \dots, n_k]. \end{array}$$

Proof. Note that if $\frac{r}{s} = [n_1, n_2, \dots, n_k]$ then $\frac{r}{s} = [1, 1, 1, n_1, n_2, \dots, n_k]$ by moves (CF 1) and (CF 2). Thus we may assume k is odd or even as needed.

Case 1. We proceed by induction on odd integers.

If $k = 1$, then we are done as $W = B^{n_1} = \begin{pmatrix} 1 & 0 \\ n_1 & 1 \end{pmatrix}$ and $\frac{1}{n_1} = [n_1]$ and $\frac{0}{1} = [\emptyset]$.

Assume $W = \pm B^{n_3} \ldots A^{n_{k-1}} B^{n_k} = \begin{pmatrix} x & t \\ y & u \end{pmatrix}$ where $\frac{x}{y} = [n_3, \ldots, n_{k-1}, n_k]$ and $\frac{t}{u} = [n_3, \ldots, n_{k-1}]$. Consider

$$B^{n_1} A^{n_2} = \begin{pmatrix} 1 & -n_2 \\ n_1 & 1 - n_1 n_2 \end{pmatrix}.$$

Then

$$\begin{aligned} B^{n_1} A^{n_2} W &= \begin{pmatrix} 1 & -n_2 \\ n_1 & 1 - n_1 n_2 \end{pmatrix} \begin{pmatrix} x & t \\ y & u \end{pmatrix} \\ &= \begin{pmatrix} x - n_2 y & t - n_2 y \\ n_1 x + y - n_1 n_2 y & n_1 t + u - n_1 n_2 u \end{pmatrix}. \end{aligned}$$

Thus

$$\begin{aligned} \frac{x - n_2 y}{n_1 x + y - n_1 n_2 y} &= \frac{1}{\frac{n_1(x - n_2 y) - -y}{x - n_2 y}} = \frac{1}{n_1 - \frac{-y}{x - n_2 y}} \\ &= \cfrac{1}{n_1 - \cfrac{1}{\frac{n_2 y - x}{y}}} = \cfrac{1}{n_1 - \cfrac{1}{n_2 - \frac{x}{y}}} \\ &= \cfrac{1}{n_1 - \cfrac{1}{n_2 - [n_3, \ldots, n_k]}} = [n_1, n_2, n_3, \ldots, n_k]. \end{aligned}$$

Similarly $\frac{t - n_2 u}{n_1 t + u - n_1 n_2 u} = [n_1, n_2, n_3, \ldots, n_{k-1}]$.

Case 2. Since $[n_1] = [1, 1, 1, n_1]$, we proceed by induction on even integers.

If $k = 2$ then $W = B^{n_1} A^{n_2} = \begin{pmatrix} 1 & -n_2 \\ n_1 & 1 - n_1 n_2 \end{pmatrix}$. We are then done since

$$\frac{-n_2}{1 - n_1 n_2} = \cfrac{1}{n_1 - \cfrac{1}{n_2}} = [n_1, n_2],$$

and $1/n_1 = [n_1]$.

When $k > 2$, the conclusion follows as in case 1.

The lemma then follows. $\square$

Lemma 2.3.6. *Let* $W = \begin{pmatrix} x & t \\ y & u \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ *and* $\frac{x}{y} = [n_1, n_2, \ldots, n_k]$. *Then depending on the parity of* k,

$$W = \pm B^{n_1} A^{n_2} \ldots B^{n_k} A^N \, or \ \pm BABA^{n_1} B^{n_2} \ldots B^{n_k} A^N$$

for some $N \in \mathbb{Z}$.

Proof. Let $W_0 = \begin{pmatrix} x & t_0 \\ y & u_0 \end{pmatrix}$ where $\frac{t_0}{u_0}$ is in lowest terms with continued fraction expansion $[n_1, n_2, \ldots, n_{k-1}]$. Then since $W_0 \in \mathrm{SL}_2(\mathbb{Z})$,

$$\begin{aligned} W_0^{-1} W &= \begin{pmatrix} u_0 & -t_0 \\ -y & x \end{pmatrix} \begin{pmatrix} x & t \\ y & u \end{pmatrix} \\ &= \begin{pmatrix} 1 & tu_0 - ut_0 \\ 0 & 1 \end{pmatrix} = A^{ut_0 - tu_0}. \end{aligned}$$

Thus $W = W_0 A^N$ where $N = ut_0 - tu_0$. Then by applying Lemma 2.3.5 to W_0 we are done. $\square$

Definition 2.3.9. Assume $W \in \mathrm{SL}_2(\mathbb{Z})$ may be expressed as

$$W = P^J C^{n_k} \ldots B^{n_2} A^{n_1}$$

where $|n_i| \geq 2$ for $i = 2, \ldots, k$ such that if k is odd $J \in \{0, 2\}$ and $C = B$, and if k is even $J \in \{-1, +1\}$ and $C = A$. We call such an expression for W a *special form*.

Remark 2.3.5. Contrast this with the special form of [6].

2.3.3 Curves on the Once-Punctured Torus

Definition 2.3.10. Let $\mathcal{K}$ be the collection of essential simple closed curves on T.

Definition 2.3.11. If $K \in \mathcal{K}$, then for some choice of orientation on K, $[K] = p[a] + q[b] \in H_1(T; \mathbb{Z})$ for some coprime p and q in $\mathbb{Z}$. Then $\frac{p}{q}$ is called the *slope* of K (with respect to the choice of basis curves a and b on T).

Lemma 2.3.7. *Given K, p, and q as above, $\frac{p}{q}$ has a continued fraction expansion $[r_n, \ldots, r_2, r_1]$ of odd length if and only if*

$$K = \beta^{r_n} \circ \cdots \circ \alpha^{r_2} \circ \beta^{r_1}(a).$$

Proof. Let $W \in \mathrm{SL}_2(\mathbb{Z})$ be the change of basis matrix $W = \begin{pmatrix} p & p' \\ q & q' \end{pmatrix}$ where $\frac{p'}{q'} = [r_n, \ldots, r_2]$. Since n is odd, by Lemma 2.3.5, $W = \pm B^{r_n} \ldots A^{r_2} B^{r_1}$

Via the correspondence between $H_1(T)$ and homeomorphisms of T, we have that up to an orientation on K

$$K = \beta^{r_n} \circ \cdots \circ \alpha^{r_2} \circ \beta^{r_1} \circ \alpha^N(a),$$

since $[K] = W[a]$. Because $\alpha^N(a) = a$,

$$K = \beta^{r_n} \circ \cdots \circ \alpha^{r_2} \circ \beta^{r_1}(a).$$

These steps all reverse for the other implication. □

Remark 2.3.6. If

$$K = \alpha^{r_n} \circ \cdots \circ \alpha^{r_2} \circ \beta^{r_1}(a),$$

then

$$K = \beta^0 \circ \alpha^{r_n} \circ \cdots \circ \alpha^{r_2} \circ \beta^{r_1}(a).$$

Hence the slope of K has the continued fraction expansion $[0, r_n, \ldots, r_2, r_1]$.

2.4 Lens Spaces and Berge Knots

2.4.1 Lens Spaces

Consider the unknot in S^3, and let $\frac{p}{q}$ be a slope in the standard basis.

Definition 2.4.1. A *lens space* is the 3-manifold $L(p,q) \cong X_{\text{Unknot}}(\frac{p}{q})$.

Notice that a lens space is the union of two solid tori identified along their boundaries. Typically we exclude the special cases $S^3 \cong L(1,n)$ for $n \in \mathbb{Z}$ and $S^1 \times S^2 \cong L(0,1)$ from being called lens spaces.

Definition 2.4.2. If K is a knot in S^3, then we say K *admits a lens space surgery* if $X_K(\frac{p}{q})$ is a lens space for some slope $\frac{p}{q}$.

2.4.2 Berge Knots

See Berge's paper [3] for a complete description of this collection of knots.

Let H_1 be a standardly embedded genus 2 handlebody in S^3, and let H_2 be its complementary (genus 2) handlebody. Set $\Sigma = \partial H_1 = \partial H_2$, and let K be a simple closed curve on Σ.

Definition 2.4.3. If K represents a primitive element of $\pi_1(H_i) \cong \mathbb{Z} * \mathbb{Z}$ for both $i = 1, 2$, then we say K is a *double primitive* curve on Σ.

Recall that if a is a primitive element of the group $\mathbb{Z} * \mathbb{Z}$, then there exists an element $b \in \mathbb{Z} * \mathbb{Z}$ such that $\langle a, b \rangle \cong \mathbb{Z} * \mathbb{Z}$. The condition that K represents a primitive element in $\pi_1(H_i)$ is equivalent to the existence of a properly embedded disk D_i in H_i so that $|K \cap \partial D_i| = 1$, $i = 1, 2$.

Theorem 2.4.1. (Berge [3]) *If K is a double primitive curve on Σ, then K admits a lens space surgery.*

Definition 2.4.4. The knots that may be written as double primitive curves on Σ are referred to as *Berge knots*.

2.4.3 Knots on a Genus One Fiber

Definition 2.4.5. Let $\mathcal{K}_\eta$ be the image of $\mathcal{K}$ under the inclusion

$$T \to T \times \{0\} \subset T \times I/_\eta = M_\eta.$$

If $K \in \mathcal{K}_\eta$ is a knot in the once-punctured torus bundle M_η, then we say that K is a *level knot* .

Note that a level knot $K \subset M_\eta$ is "level" in the sense that K is contained in a level set of the projection $p : M_\eta \to S^1$ where each fiber is the preimage of a point.

When $T \times I/_\eta \subset S^3$ is a knot complement (i.e. the complement of a trefoil or the Figure Eight Knot), we also let $\mathcal{K}_\eta$ be the image of $\mathcal{K}$ under the further inclusion $T \to T \times \{0\} \subset T \times I/_\eta \subset S^3$. In these cases, $\mathcal{K}_\eta$ is the family of knots on the fiber of a trefoil or the Figure Eight Knot. These two families are Berge's families VII) and VIII) in [3] respectively. Thus the knots in $\mathcal{K}_\eta$ are Berge knots and hence have lens space surgeries. These knots are the primary focus of this dissertation.

Remark 2.4.1. Berge describes his knots up to reflection. The knots in Berge's family VII) are those that lie on the fiber of the Right Handed Trefoil. We however typically work with the family of knots that lie on the fiber of the Left Handed Trefoil and obtain the family of knots that lie on the Right Handed Trefoil by reflection. When setup as in the beginning of the following chapter, our knots of slope $\frac{x}{y}$ on the fiber of the Left Handed Trefoil correspond by reflection to Berge's knots $[k] = -x[g_1] + y[g_2]$ of his family VII). Note this difference when using Berge's calculation of the lens spaces yielded by surgery on these knots.

Chapter 3

Surgery Descriptions of Knots

3.1 Surgery Description of Knots on a Genus One Fiber

Recall (§2.2.1) T is a once-punctured torus and a and b are oriented simple closed curves on T that intersect once as in Figure 2.5. Assume T is oriented so that $a \cdot b = +1$.

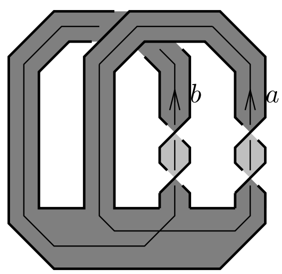


Figure 3.1: Left Handed Trefoil with fiber and basis.

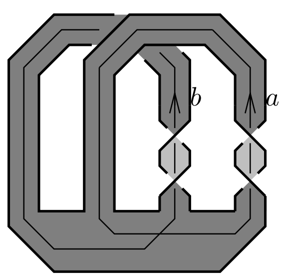


Figure 3.2: Figure Eight Knot with fiber and basis.

Also recall the construction in §2.2.2 of a once-punctured torus bundle $T \times I/_\eta$ where $I = [0,1]$ and $(x,1) \sim (\eta(x),0)$. Equivalently, $(\eta^{-1}(x),1) \sim (x,0)$. In this chapter let us view once-punctured torus bundles slightly differently. Let M_η denote the fiber bundle $T \times [-n, n+1]/\eta$ where $(\eta^{-1}(x), n+1) \sim (x, -n)$ for some $n \in \mathbb{N}$ and some homeomorphism $\eta : T \to T$. If $\eta^{-1} = \beta \circ \alpha$, then M_η may be recognized as the complement of the Left Handed Trefoil depicted in Figure 3.1 (as the boundary of the plumbing of two negative Hopf bands).

If $\eta^{-1} = \alpha^{-1} \circ \beta^{-1}$, then M_η is the complement of the Right Handed Trefoil. The Right Handed Trefoil may be obtained from the Left Handed Trefoil by reflecting Figure 3.1 through the plane of the page. Consequentially, the complement of the Right Handed Trefoil is that of the Left Handed Trefoil with opposite orientation. For this reason we will not be expressly dealing with the Right Handed Trefoil.

If $\eta^{-1} = \beta \circ \alpha^{-1}$, then M_η is the complement of the Figure Eight Knot depicted in Figure 3.2 (as the boundary of the plumbing of a positive Hopf band onto a negative Hopf band). Note that in all cases, $M_\eta \subset S^3$.

Throughout the remainder of this chapter, η^{-1} is assumed to be either $\beta \circ \alpha$ or $\beta \circ \alpha^{-1}$.

3.1.1 The link $L(2n+1, \eta)$.

Consider $M_\eta = T \times [-n, n+1]/_\eta \subset S^3$. For $n \in \mathbb{N}$ and $i \in \{-n, -n+1, \ldots, n\}$, set $L_i = a \times \{i\}$ or $L_i = b \times \{i\}$ if i is even or odd respectively.

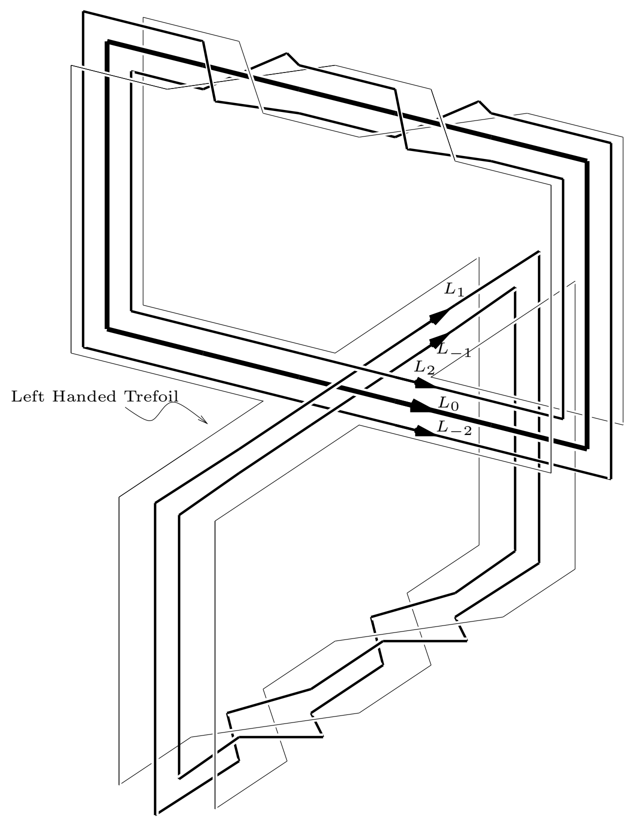


Figure 3.3: The link $L(5)$ and the Left Handed Trefoil.

Definition 3.1.1. Define $L(2n+1,\eta)$ to be the oriented link

$$\bigcup_{i=-n}^{n} L_i \subset M_\eta \subset S^3.$$

We write $L(2n+1)$ when η is understood.

Figure 3.3 shows the link $L(5)$ where $\eta^{-1} = \beta\circ\alpha$. Included in the figure for reference is the corresponding genus one fibered knot, the Left Handed Trefoil.

Consider the exterior of $L(2n+1)$, $X_{L(2n+1)}$. For each $i \in \{-n,\ldots,n\}$, let μ_i be the standard meridian for $\partial N(L_i)$, and let λ_i be the longitude associated to the isotopy class of a component of $\partial N(L_i) \cap T \times \{i\}$.

3.1.2 Obtaining $K \in \mathcal{K}_\eta$ by surgery on $L(2n+1, \eta)$

Notice that λ_i is not the standard longitude for L_i. Moreover, for $i > 0$, the representatives of λ_i and λ_{-i} together bound an annulus R_i in $X_{L_i \cup L_{-i}}$, the exterior of $L_i \cup L_{-i}$, that intersects $T \times \{0\}$ along a curve isotopic to either $b \times \{0\}$ if i is odd or $a \times \{0\}$ if i is even. Therefore, using the bases $\{\mu_i, \lambda_i\}$ and $\{\mu_{-i}, \lambda_{-i}\}$, the surgery $X_{L_{-i} \cup L_i}(-\frac{1}{r_i}, \frac{1}{r_i})$ realizes a "spinning" along R_i. On $T \times \{0\}$ this restricts to the Dehn twist α^{r_i} or β^{r_i} depending on the parity of i. (C.f. Remark 2.1.4)

Proposition 3.1.1. *If $K \in \mathcal{K}_\eta$, then K may be obtained by surgery on $L(2n+1, \eta)$ for some $n \in \mathbb{N}$.*

Proof. Express the knot $K \subset T \times \{0\}$ as the image of a composition of Dehn twists:

$$K = \gamma^{r_n} \circ \cdots \circ \alpha^{r_2} \circ \beta^{r_1}(a)$$

where γ is α or β depending on the parity of n. (This may be done via Lemma 2.3.7 or its following Remark 2.3.6.)

By nesting the surgery realization of Dehn twists, we may obtain K by surgery on $L(2n+1)$. Let $\rho_i = \frac{1}{r_i}$ or $-\frac{1}{r_i}$ if i is positive or negative respectively. Set $\rho_0 = *$ to indicate that Dehn surgery is not done on L_0. Let $\overline{\rho} = (\rho_{-n}, \ldots, \rho_n)$. Then K is the image of L_0 in $X_{L(2n+1)}(\overline{\rho}) = S^3$. $\square$

3.2 Volumes

Here we address the main theorem:

Theorem 3.2.1. *The set of Berge knots include hyperbolic knots of arbitrarily large volume.*

An immediate corollary is:

Corollary 3.2.2. *All Berge knots cannot be obtained as surgery on a single link.*

See Appendix A for more about surgery descriptions of Berge knots.

To prove Theorem 3.2.1, we will use Corollary 3.3.4 which states that $L(2n+1, \eta)$ is a hyperbolic link for $n > 1$. We defer this corollary and its proof until the end of this chapter.

Proof. (of Theorem 3.2.1) The classes of Berge knots we will exploit here are those with representatives in $\mathcal{K}_\eta$ where η^{-1} is $\beta \circ \alpha$, $\alpha^{-1} \circ \beta^{-1}$, or $\beta \circ \alpha^{-1}$ corresponding to knots which lie on the fiber of the Left Handed Trefoil, Right Handed Trefoil, or Figure Eight Knot respectively. (Note that the knots on the fiber of the Right Handed Trefoil are reflections of knots on the Left Handed Trefoil.) By Proposition 3.1.1, we may obtain these knots as surgeries on $L(2n+1, \eta)$ according to their description as a product of Dehn twists.

By Corollary 3.3.4, $X_{L(2n+1)}$ is a hyperbolic manifold with $2n+1$ cusps for $n \geq 2$. Adams [1] shows that a hyperbolic manifold with N cusps has volume at least $v_N \geq n \cdot v$. Here v_N is the volume of the smallest volume N cusped hyperbolic manifold, and v is the volume of a regular ideal tetrahedron. Thus $\mathrm{vol}(X_{L(2n+1)}) \geq (2n+1) \cdot v$. Thurston's Hyperbolic Dehn Surgery Theorem [20] states that large enough filling on a cusp of a (finite volume) hyperbolic manifold will yield a hyperbolic manifold with volume near that of the unfilled manifold. Thus given $\varepsilon > 0$ we may choose $|r_1|, \ldots, |r_n| \gg 0$ so that if $K \in \mathcal{K}_\eta$ corresponds to the continued fraction expansion $[r_1, r_2, \ldots, r_k]$ as in Lemma 2.3.7 and $\overline{\rho}$ is chosen as in Proposition 3.1.1, then

$$\mathrm{vol}(K) = \mathrm{vol}(X_{L(2n+1)}(\overline{\rho}) - N(L_0)) > \mathrm{vol}(X_{L(2n+1)}) - \varepsilon.$$

Hence $\mathrm{vol}(K) > (2n+1) \cdot v - \varepsilon$.

For each $n > 1$ we may thus choose as sequence of knots $\{K_n\}_{n=2}^{\infty} \subset \mathcal{K}_\eta$ with corresponding continued fraction expansions $[r_1(n), \ldots, r_n(n)]$ where for each n the $|r_i(n)|$ are sufficiently large so that $\mathrm{vol}(K_n) \to \infty$ as $n \to \infty$. $\square$

3.3 Homeomorphism of Link Exteriors

Definition 3.3.1. For $n \in \mathbb{N}$, let $C(2n+1) = \bigcup_{i=1}^{n} C_i$ be the *minimally twisted chain link of* $2n+1$ *components* as shown for each n even and n odd in Figure 3.4. Note there are actually two such links, $C(2n+1)$ and its reflection $-C(2n+1)$, depending on how the clasping of C_n and C_{-n} is done. For each $i \in \{-n, \ldots, n\}$, let $\{m_i, l_i\}$ be the standard meridian-longitude pair for C_i.

In this section we will show

Theorem 3.3.1. *The links* $L(2n+1, \eta)$ *and* $C(2n+1)$ *have homeomorphic exteriors. I.e.* $X_{L(2n+1,\eta)} \cong X_{C(2n+1)}$.

Consequentially, we will use this homeomorphism to obtain descriptions of the knots $K \in \mathcal{K}_\eta$ as surgeries on these chain links. Through different methods Yuichi Yamada [22] has also obtained descriptions of many of these

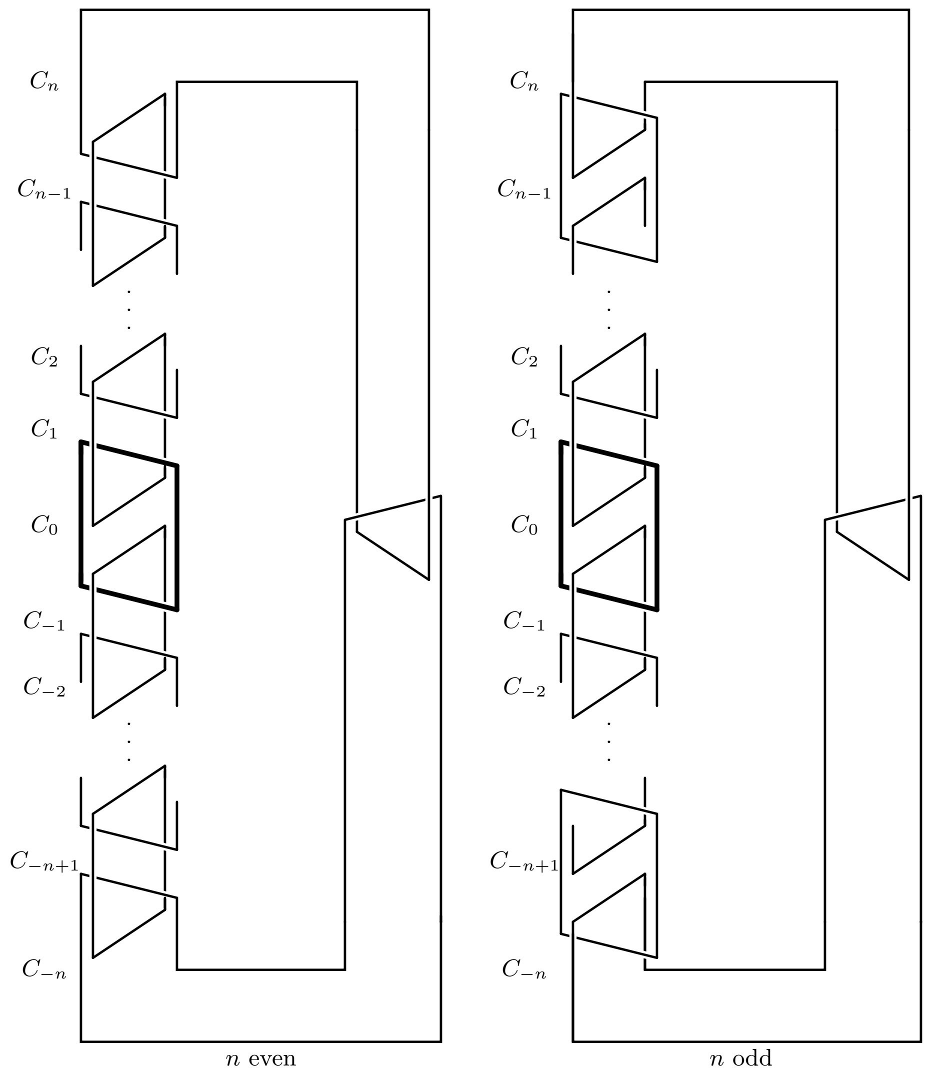


Figure 3.4: The minimally twisted $2n + 1$ chain for n even and n odd.

knots as surgeries on minimally twisted chain links. In Appendix A we describe the remaining Berge knots as surgeries on the minimally twisted five chain link, MT5C $= C(5)$. Furthermore, though it may be done directly, this homeomorphism simplifies the proof of the hyperbolicity of $L(2n + 1)$.

3.3.1 Proof of Theorem 3.3.1

The following proof employs tangles. See Section 2.1.2 for the relevant definitions.

Throughout this subsection and the next, the choice of $\epsilon = \pm 1$ depends on the choice of η. This is encapsulated by

$$\eta^{-1} = \beta \circ \alpha^{\epsilon}$$

so that $\epsilon = +1$ corresponds to the Left Handed Trefoil and $\epsilon = -1$ corresponds to the Figure Eight Knot. Furthermore, in the ensuing figures, blocks with $\frac{1}{2}\epsilon$ indicate the half twists as shown in Figure 3.5.

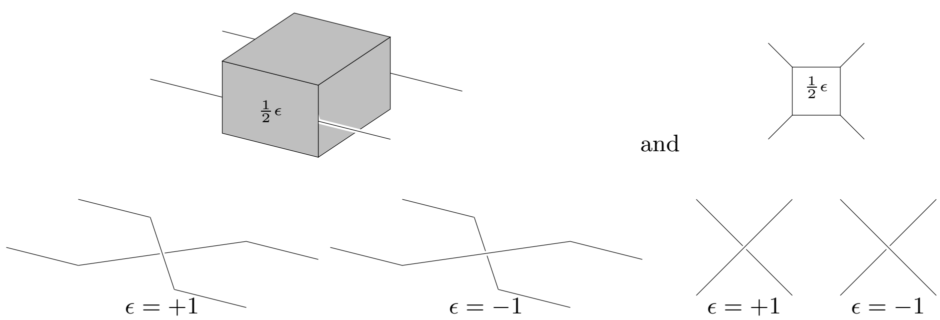


Figure 3.5: Twistings of $\frac{1}{2}\epsilon$ for $\epsilon = \pm 1$.

Proof. The links $L(2n+1, \eta)$ all admit a strong involution as shown in Figure 3.6 for $n = 2$. The genus one fibered knot is included for reference as it is invariant under the involution, too. Quotienting the link exterior $X_{L(2n+1,\eta)}$ by the involution yields the tangle (B_{2n+1}, t_L) where B_{2n+1} is a $2n+1$ punctured S^3 and t_L is the image of the involution axis. Hence the double branched cover of B_{2n+1} branched over t_L is $X_{L(2n+1,\eta)}$. A sequence of isotopies of (B_{2n+1}, t_L) for $n = 2$ into a "nice" form are shown in Figures 3.7 (a), (b), and (c), and

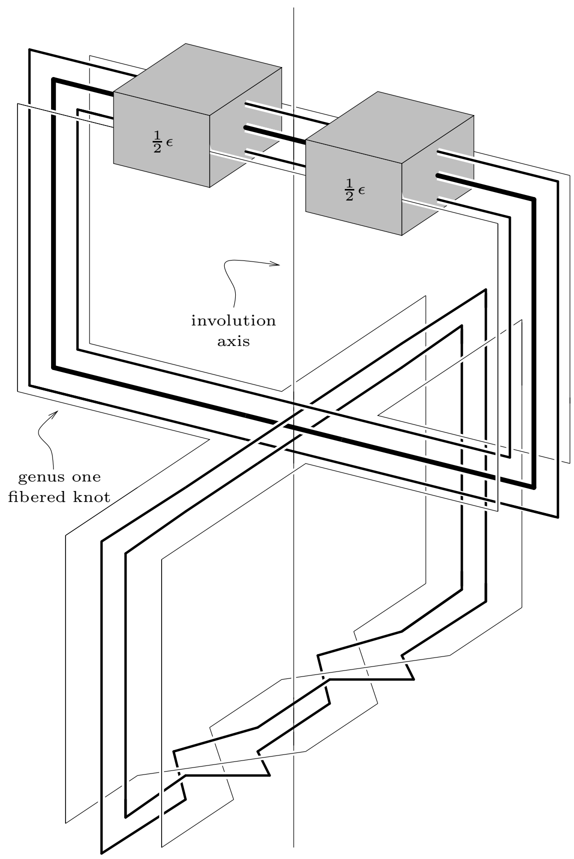


Figure 3.6: The link $L(5, \eta)$ with an axis of strong involution and the genus one fibered knot.

are continued in Figures 3.8 (a) and (b) with each choice of ϵ. The quotient of the genus one fibered knot is also shown in these figures.

We may trace the longitudes λ_i and standard meridians μ_i of $\partial N(L_i)$ through the quotient of $X_{L(2n+1,\eta)}$ and its subsequent isotopies as in Figures 3.6, 3.7, and 3.8 to obtain the corresponding framings $\hat{\lambda}_i$ and $\hat{\mu}_i$ on the ith boundary components $\widehat{\partial N(L_i)}$ of $\partial(B_{2n+1}, t_L)$. The picture of (B_{2n+1}, t_L) for general n with framings is shown in Figure 3.9.

The minimally twisted chain links $C(2n+1)$ all admit strong involutions

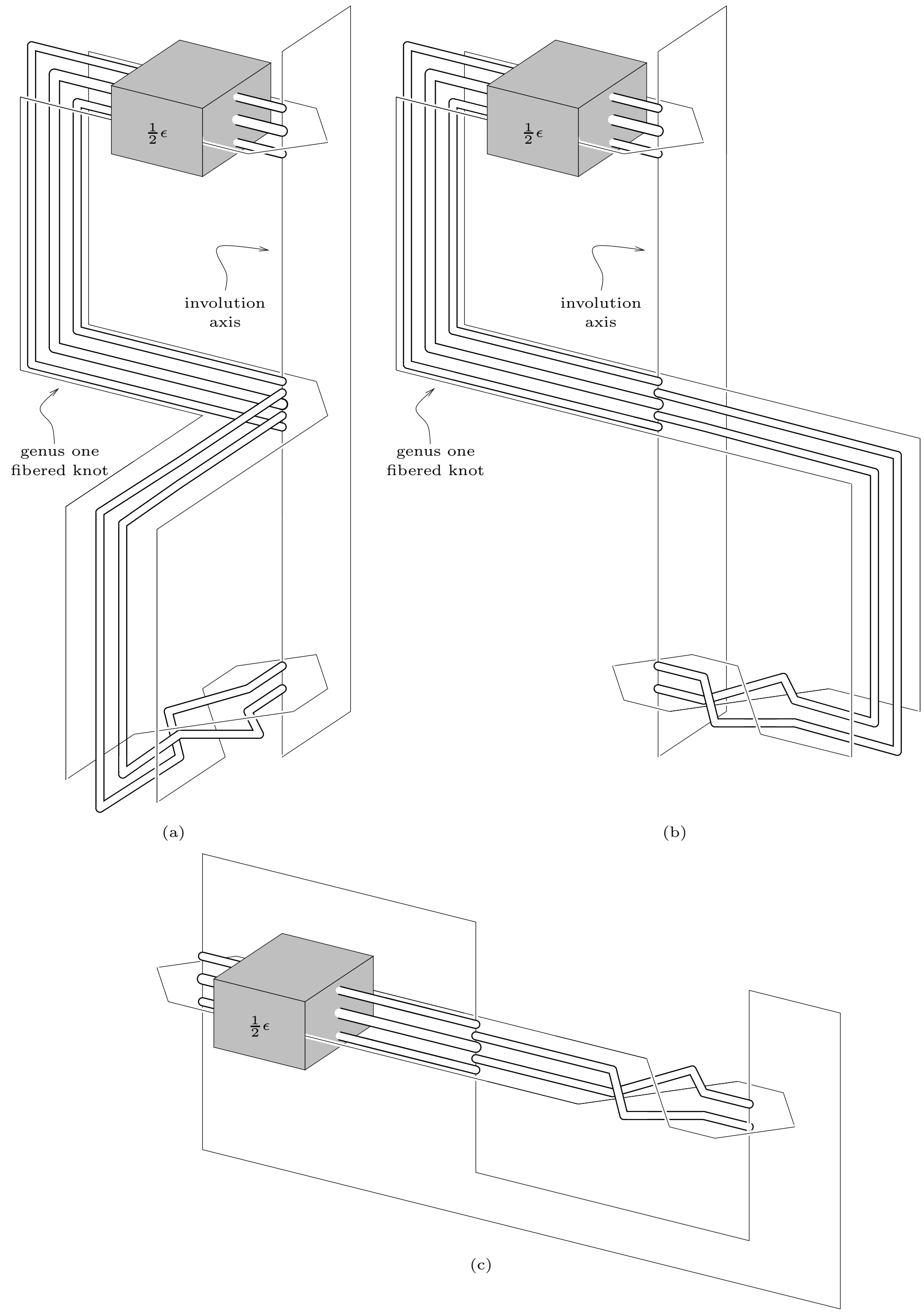


Figure 3.7: The quotient of the complement of the link $L(5, \eta)$ by the strong involution, (B_5, t_L), followed by isotopies.

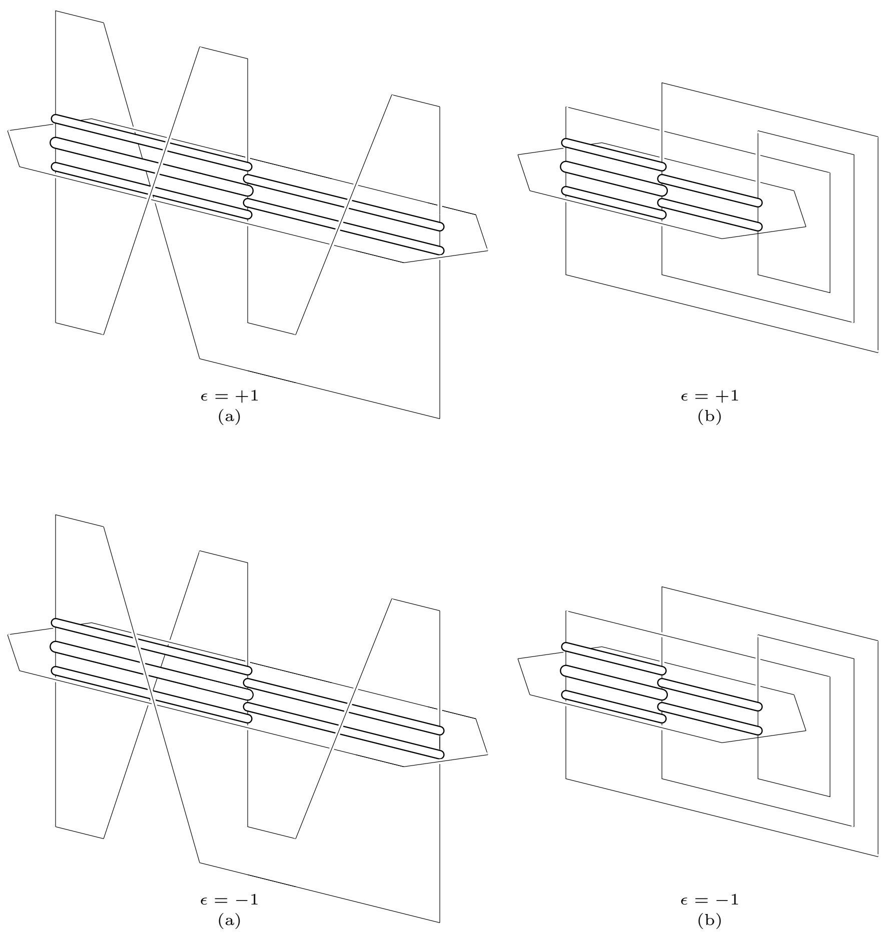


Figure 3.8: Further isotopies of (B_5, t_L).

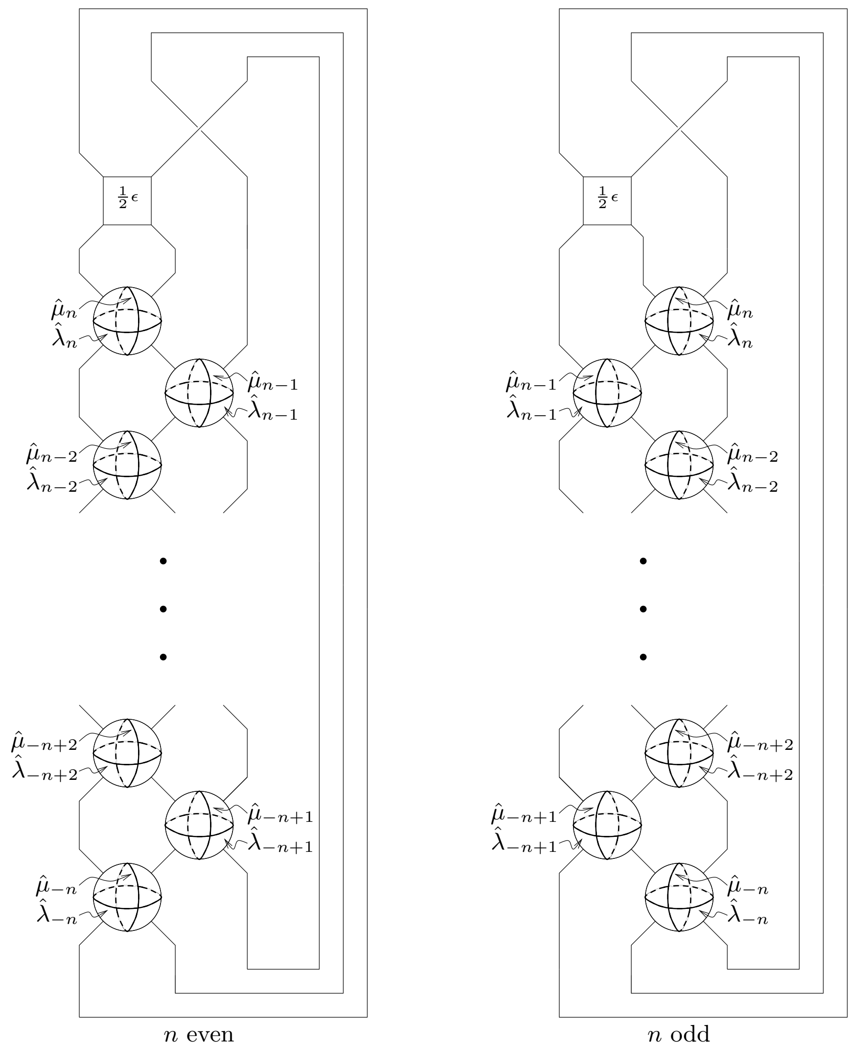


Figure 3.9: The tangles (B_{2n+1}, t_L) for each n even and odd.

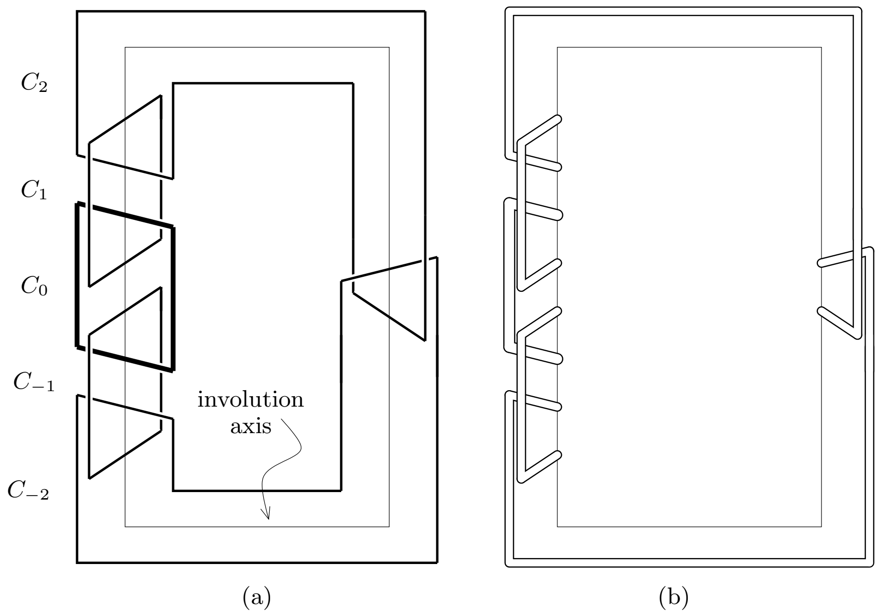


Figure 3.10: The chain link $C(5)$ with an axis of strong involution and the quotient of its complement, (B_5, t_C).

too as shown in Figure 3.10 (a) for $n = 2$. Quotienting the chain link exterior $X_{C(2n+1)}$ by the involution yields the tangle (B_{2n+1}, t_C) where B_{2n+1} is a $2n+1$ punctured S^3 and t_C is the image of the involution axis. Hence the double branched cover of B_{2n+1} branched over t_C is $X_{C(2n+1)}$. Isotopies of (B_{2n+1}, t_C) for $n = 2$ into a "nice" form are shown in Figures 3.10 (b) and 3.11 (a) and (b).

We may trace the standard longitudes l_i and meridians m_i for $\partial N(C_i)$ through the quotient of $X_{C(2n+1)}$ and the subsequent isotopies as in Figures 3.10 and 3.11 to obtain the corresponding framings $\hat{l}_i$ and $\hat{m}_i$ on the ith boundary components $\widehat{\partial N(C_i)}$ of $\partial(B_{2n+1}, t_C)$. The picture of (B_{2n+1}, t_C) for general n with framings is shown in Figure 3.12.

Case 1: n is even.

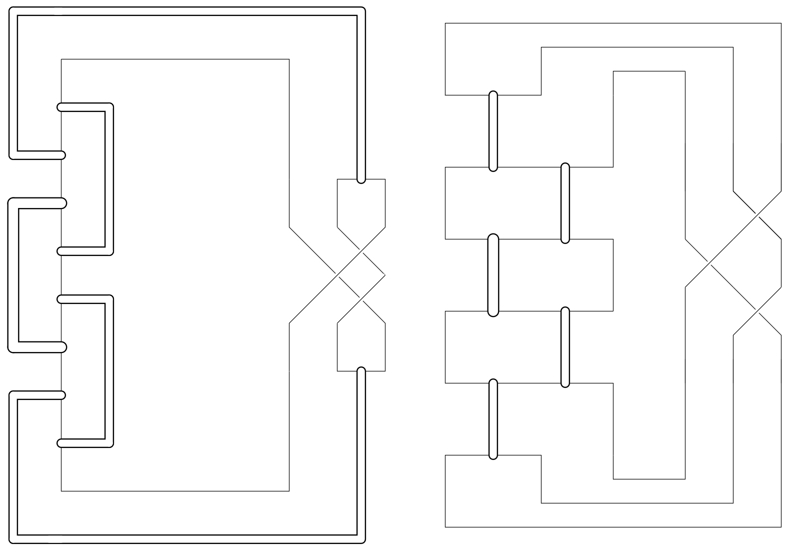


Figure 3.11: Isotopies of (B_5, t_C).

Figure 3.13 indicates a homeomorphism $\hat{h}$ between the two tangles (B_{2n+1}, t_L) and (B_{2n+1}, t_C). The nth and $-n$th boundary components of (B_{2n+1}, t_L) may absorb the extra twists into their framings as shown in Figure 3.14 to complete the homeomorphism.

The homeomorphism

$$\hat{h}\colon (B_{2n+1}, t_L) \to (B_{2n+1}, t_C)$$

then lifts to a homeomorphism

$$h\colon X_{L(2n+1,\eta)} \to X_{C(2n+1)}$$

of the double branched covers.

Case 2: n is odd.

The result of further isotopies of the tangles (B_{2n+1}, t_L) and (B_{2n+1}, t_C) as depicted in Figures 3.9 and 3.12 for n odd are shown together in Figure 3.15

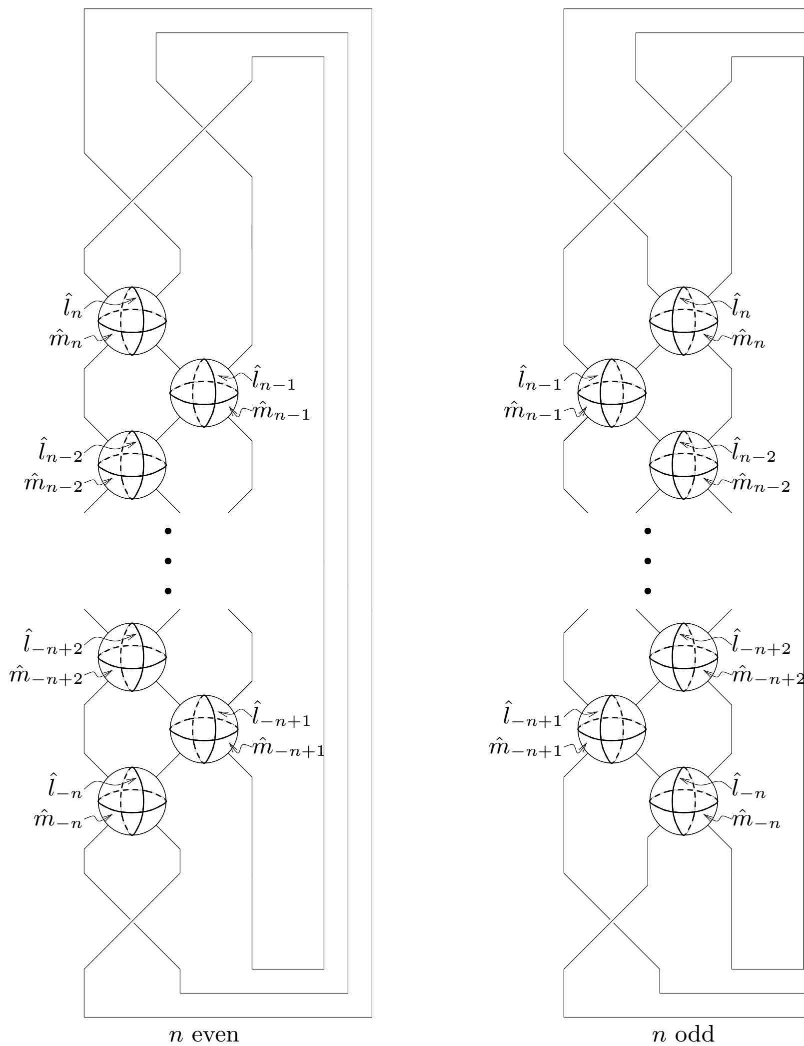


Figure 3.12: The tangles (B_{2n+1}, t_C) for each n even and odd.

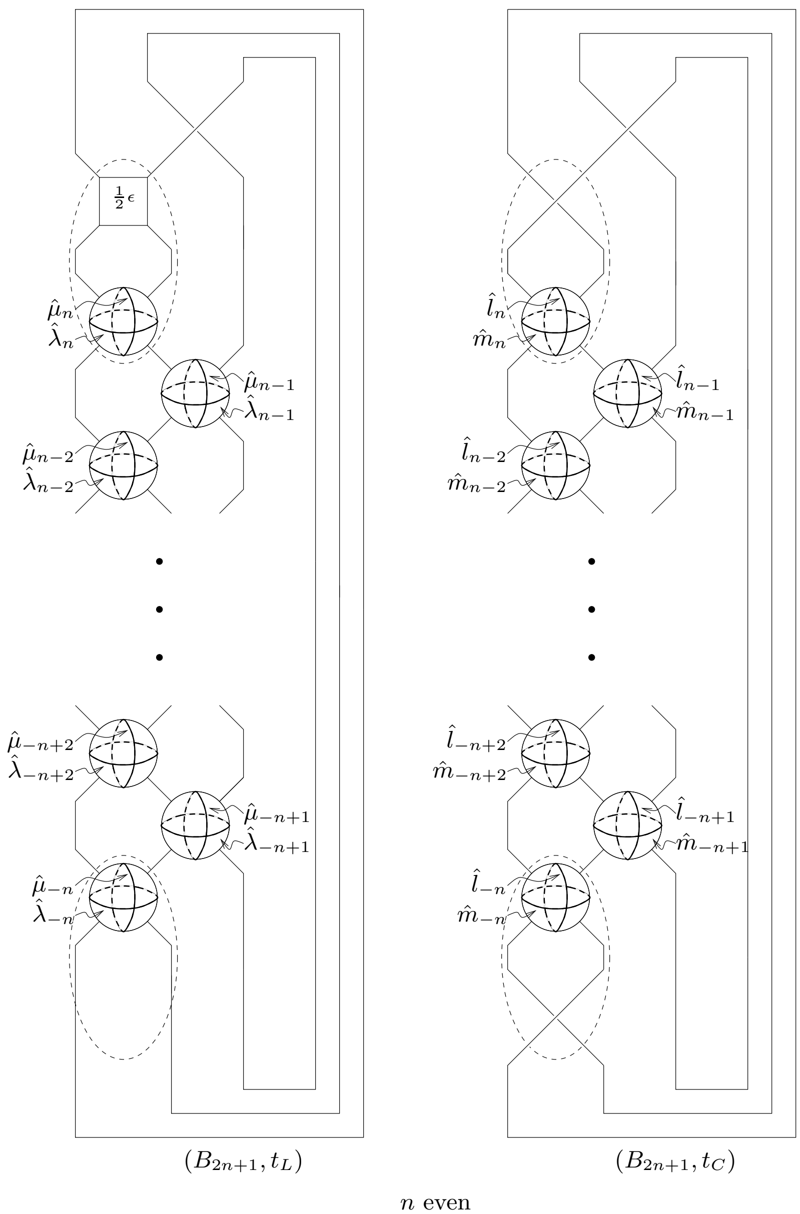


Figure 3.13: The tangles (B_{2n+1}, t_L) and (B_{2n+1}, t_C) for n even.

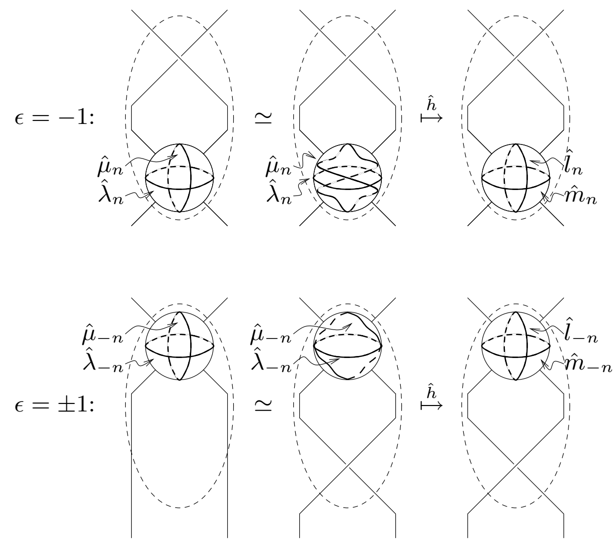


Figure 3.14: Twistings of the nth and $-n$th boundary components as needed for $\hat{h}$ for n even.

suggesting a homeomorphism $\hat{h}$. The isotopy of (B_{2n+1}, t_C) employs the "braid relation move" of Figure 3.16.

If $\epsilon = +1$, the nth boundary component may absorb an extra twist into its framing as shown in Figure 3.17 to complete the homeomorphism $\hat{h}$ between (B_{2n+1}, t_L) and (B_{2n+1}, t_C) indicated in Figure 3.15.

The homeomorphism

$$\hat{h}\colon (B_{2n+1}, t_L) \to (B_{2n+1}, t_C)$$

then lifts to a homeomorphism

$$h\colon X_{L(2n+1,\eta)} \to X_{C(2n+1)}$$

of the double branched covers.

If $\epsilon = -1$, take the reflection of (B_{2n+1}, t_C) through the plane of the page. The resulting tangle (B_{2n+1}, t_{-C}) has the complement of $-C(2n+1)$

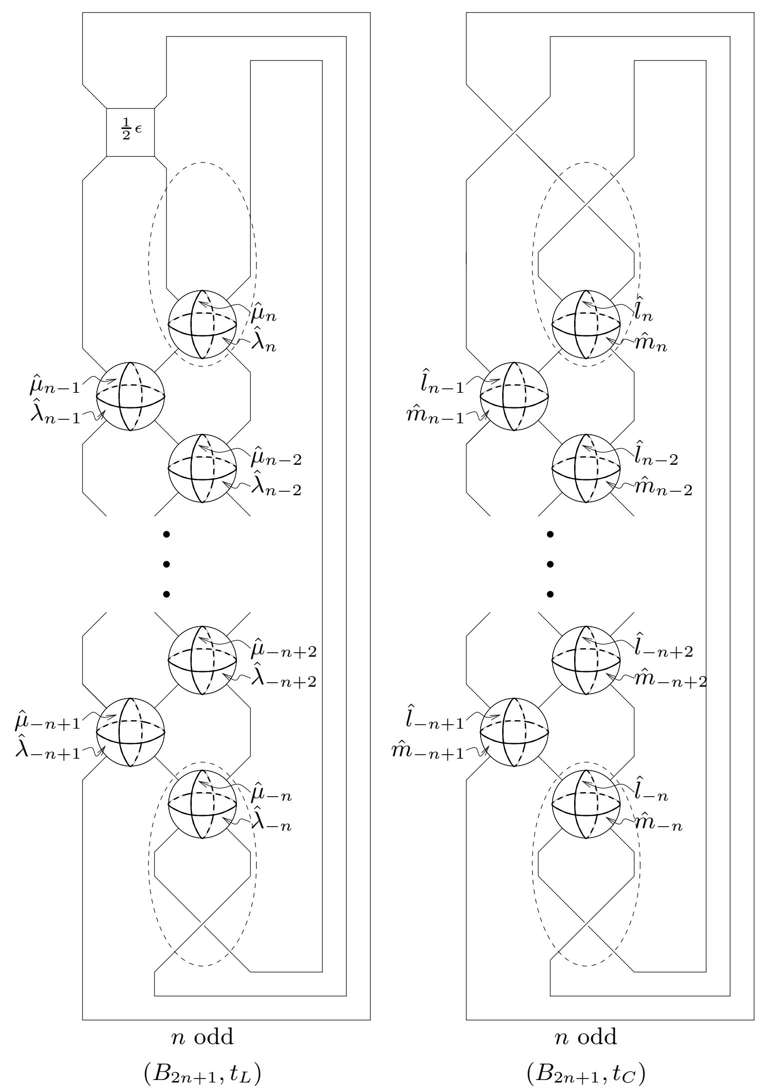


Figure 3.15: The tangles (B_{2n+1}, t_L) and (B_{2n+1}, t_C) for n odd.

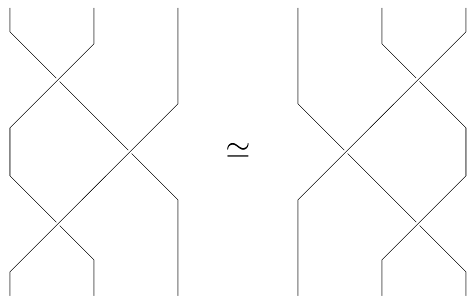


Figure 3.16: A braid isotopy.

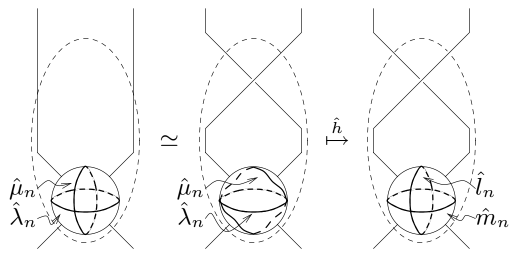


Figure 3.17: Twisting of the nth boundary component as needed for $\hat{h}$ for n odd and $\epsilon = +1$.

as its double branched cover. The nth and $-n$th boundary components of (B_{2n+1}, t_L) may absorb the extra twists into their framings as shown in Figure 3.18 to complete the homeomorphism $\hat{h}$ between (B_{2n+1}, t_L) and (B_{2n+1}, t_{-C}).

The homeomorphism

$$\hat{h}\colon (B_{2n+1}, t_L) \to (B_{2n+1}, t_{-C})$$

then lifts to a homeomorphism

$$h\colon X_{L(2n+1,\eta)} \to X_{-C(2n+1)}$$

of the double branched covers.

$\square$

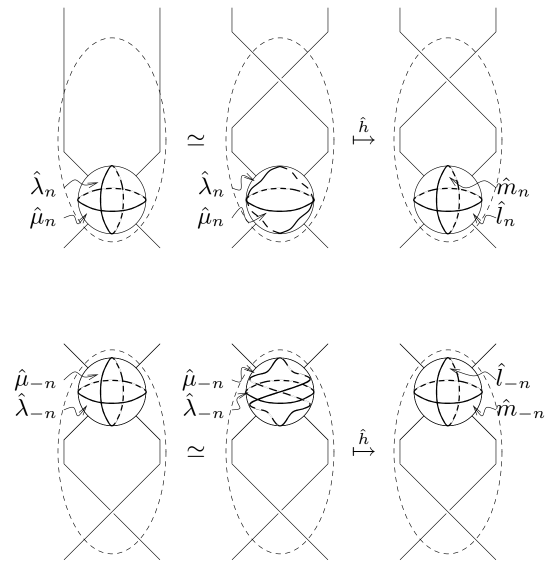


Figure 3.18: Twisting of the nth and $-n$th boundary components as needed for $\hat{h}$ for n odd and $\epsilon = -1$.

3.3.2 Surgeries on chain links.

The homeomorphisms $\hat{h}$ in the proof of Theorem 3.3.1 describe how the framed boundary components of (B_{2n+1}, t_L) map to the framed boundary components of $(B_{2n+1}, t_{\pm C})$. In the lift to the double branched covers, this confers how the homeomorphism h maps the boundary components of $X_{L(2n+1,\eta)}$ to the boundary components of $X_{\pm C(2n+1)}$ in terms of their corresponding meridian-longitude basis pairs. We thereby obtain mappings of slopes and translate surgery coefficients for $L(2n+1, \eta)$ to surgery coefficients for $\pm C(2n+1)$.

In the following maps, for each $i \in \{-n, \ldots, n\}$ the curves μ_i and λ_i on $\partial N(L_i)$ and the curves m_i and l_i on $\partial N(C_i)$ are thought of synonymously with their homological representatives so that we may add and subtract them to produce other curves on these tori. Furthermore these maps should be taken up to reversal of orientations of pairs $\{m_i, l_i\}$.

For each $\hat{h}$ and for each $i \neq \pm n$, the framing curves $\hat{\mu}_i$ and $\hat{\lambda}_i$ map to the framing curves $\hat{l}_i$ and $\hat{m}_i$ respectively. Lifting to the double branched covers,

$$\begin{array}{rcl} \mu_i & \stackrel{h}{\mapsto} & -l_i \\ \lambda_i & \stackrel{h}{\mapsto} & m_i. \end{array}$$

If n is even, then $\hat{h}$ maps the framings of the nth and $-n$th boundary components with twisting as indicated in Figure 3.14. In the double branched cover,

$$\begin{array}{rcl} \mu_{-n} + \lambda_{-n} & \stackrel{h}{\mapsto} & -l_{-n} \\ \lambda_{-n} & \stackrel{h}{\mapsto} & m_{-n} \end{array} \quad \text{and} \quad \begin{array}{rcl} \mu_n + (1-\epsilon)\lambda_n & \stackrel{h}{\mapsto} & -l_n \\ \lambda_n & \stackrel{h}{\mapsto} & m_n. \end{array}$$

If n is odd and $\epsilon = +1$, then $\hat{h}$ maps the framing of the nth boundary component with twisting as indicated in Figure 3.17. The $-n$th boundary component behaves as the others. In the double branched cover,

$$\begin{array}{rcl} \mu_{-n} & \stackrel{h}{\mapsto} & -l_{-n} \\ \lambda_{-n} & \stackrel{h}{\mapsto} & m_{-n} \end{array} \quad \text{and} \quad \begin{array}{rcl} \mu_n - \lambda_n & \stackrel{h}{\mapsto} & -l_n \\ \lambda_n & \stackrel{h}{\mapsto} & m_n. \end{array}$$

If n is odd and $\epsilon = -1$, then $\hat{h}$ maps the framings of the nth and $-n$th boundary components with twisting as indicated in Figure 3.18. In the double branched cover,

$$\begin{array}{rcl} \mu_{-n} + 2\lambda_{-n} & \stackrel{h}{\mapsto} & -l_{-n} \\ \lambda_{-n} & \stackrel{h}{\mapsto} & m_{-n} \end{array} \quad \text{and} \quad \begin{array}{rcl} \mu_n + \lambda_n & \stackrel{h}{\mapsto} & -l_n \\ \lambda_n & \stackrel{h}{\mapsto} & m_n. \end{array}$$

Therefore, given the slope vector

$$\overline{\rho} = \left((\frac{p}{q})_{-n}, (\frac{p}{q})_{-n+1}, \ldots, (\frac{p}{q})_{n-1}, (\frac{p}{q})_n \right)$$

for $L(2n+1, \eta)$ in terms of the bases $\{\mu_i, \lambda_i\}$, we may obtain the corresponding slope vector $h(\overline{\rho})$ for $\pm C(2n+1)$ in terms of the bases $\{m_i, l_i\}$ as follows.

If n is even, we obtain

$$h(\overline{\rho}) = \left((-\frac{q}{p} + 1)_{-n}, (-\frac{q}{p})_{-n+1}, \ldots, (-\frac{q}{p})_{n-1}, (-\frac{q}{p} + (1-\epsilon))_n \right)$$

for $C(2n+1)$.

If n is odd, we obtain

$$h(\overline{\rho}) = \left((-\frac{q}{p} + (1-\epsilon))_{-n}, (-\frac{q}{p})_{-n+1}, \ldots, (-\frac{q}{p})_{n-1}, (-\frac{q}{p} - \epsilon)_n \right)$$

for $\epsilon \cdot C(2n+1)$.

In particular, given $K \in \mathcal{K}_\eta$ with complement X_K described as $\overline{\rho}$-Dehn surgery on $L(2n+1,\eta)$ where

$$\overline{\rho} = \left((-\frac{1}{r_n})_{-n}, (-\frac{1}{r_{n-1}})_{-n+1}, \ldots, (-\frac{1}{r_1})_{-1}, (\emptyset)_0, (\frac{1}{r_1})_1, \ldots, (\frac{1}{r_{n-1}})_{n-1}, (\frac{1}{r_n})_n \right)$$

as in Proposition 3.1.1 (so that the coefficient $\emptyset$ indicates that its component is to be left unfilled), then for n even

$$\begin{aligned} h(\overline{\rho}) = ((r_n + 1)_{-n}, (r_{n-1})_{-n+1}, \ldots, (r_1)_{-1}, (\emptyset)_0, \\ (-r_1)_1, \ldots, (-r_{n-1})_{n-1}, (-r_n + (1-\epsilon))_n) \end{aligned}$$

and for n odd

$$\begin{aligned} h(\overline{\rho}) = ((r_n + (1-\epsilon))_{-n}, (r_{n-1})_{-n+1}, \ldots, (r_1)_{-1}, (\emptyset)_0, \\ (-r_1)_1, \ldots, (-r_{n-1})_{n-1}, (-r_n - \epsilon)_n) \,. \end{aligned}$$

Furthermore, the slope 0 on the 0th component of $X_{\pm C(2n+1)}(h(\overline{\rho})) \cong X_K$ is the meridional slope for K.

3.3.3 Hyperbolicity of the links $L(2n+1, \eta)$.

Corollary 3.3.2. (of [15], Theorem 5.1 (ii).) *The links $C(2n{+}1)$ are hyperbolic for $n > 1$.*

Proof. This is an immediate corollary of

Theorem 3.3.3. (Theorem 5.1 (ii) of Neumann-Reid [15]) *$C(p, s)$ has a hyperbolic structure (complete of finite volume) if and only if $\{|p{+}s|, |s|\} \not\subset \{0, 1, 2\}$.*

Our links $C(2n + 1)$ correspond to their links $C(2n + 1, -n - 1)$ or $C(2n + 1, -n)$ depending on how the last clasping is done. Since $n > 1$ the theorem applies. $\square$

From the above homeomorphism of link exteriors and corollary, we conclude

Corollary 3.3.4. *$L(2n + 1, \eta)$ is a hyperbolic link for $n \geq 2$.*

Chapter 4

Closed Essential Surfaces

In this chapter we develop an algorithm to list all the closed essential surfaces in the complement of any given knot in S^3 that belongs to $\mathcal{K}_\eta$. We first develop an algorithm to list all the surfaces in once-punctured torus bundle that are both disjoint from a given level knot and essential its complement. Then we determine when Dehn filling the boundary of the once-punctured torus bundle permits a surface to cap off to a closed surface which is essential in the complement of the knot. We take lead from the work *Incompressible surfaces in once-punctured torus bundles* by Culler-Jaco-Rubinstein [6] which constructs an algorithm to list surfaces as in the title and adapt their methods to our situation. We assume familiarity with this paper throughout this chapter.

4.1 Twisted Surfaces

We begin by describing a certain type of surface in once-punctured torus bundles and in the complement of a level knot in a once-punctured torus bundle. Much of the terminology and methods used here are borrowed or adapted from [6].

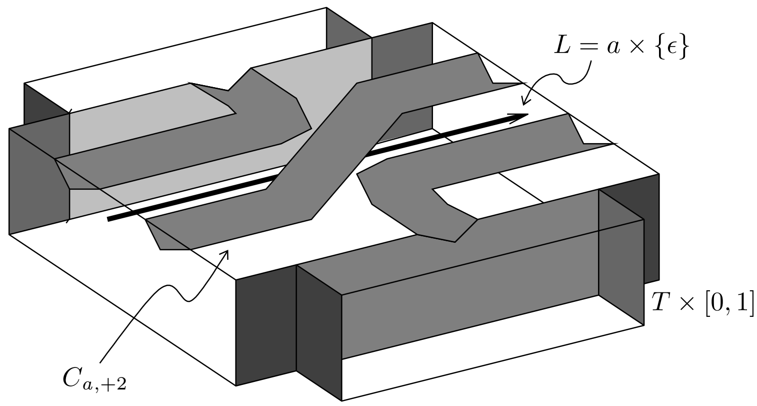


Figure 4.1: The twisted saddle $C_{a,+2}$ with level knot $a \times \{\epsilon\}$.

4.1.1 Construction of Twisted Surfaces

Here we follow parts of §2 of [6]. We describe the so-called *twisted surfaces* in a once punctured torus bundle sans a level curve and prove when they are essential.

To describe these surfaces, Culler-Jaco-Rubinstein list surfaces embedded in $T \times [0,1]$. We need only the "twisted saddle" surfaces $C_{a,n}$ and $C_{b,n}$ (c.f. §2.1 [6]). Note that the surface $C_{a,n}$ embeds in $T \times [0,1] - N(a \times \{\epsilon\})$ where $\epsilon > 0$ (so that $a \times \{\epsilon\}$ is a slight upward push-off of $a \times \{0\}$). See Figure 4.1.

As in §2.5 of [6], let $n(1), n(2), \ldots, n(k)$ be integers and $J \in \{-1, 0, 1, 2\}$ have the same parity as k. Consider the bundle

$$\begin{aligned} M &= T \times I/_{\alpha^{n(1)}} T \times I/_{\beta^{n(2)}} \ldots T \times I/_{\gamma^{n(k)}} T \times I/_{\phi^J} \\ &= T \times I/_{\phi^J \gamma^{n(k)} \ldots \beta^{n(2)} \alpha^{n(1)}} \end{aligned}$$

where

$$\gamma = \begin{cases} \beta, & \text{if } k \text{ is even,} \\ \alpha, & \text{if } k \text{ is odd.} \end{cases}$$

Note that M contains $k+1$ blocks. Take $L = a \times \{\epsilon\}$ in the first block of M for small $\epsilon > 0$. (We may actually think of L as being $a \times \{0\}$, but it is useful to have L not on a fiber along which blocks of M are glued.)

Construct the surface R by putting twisted saddles in the first k blocks of M, $C_{a,n(i)}$ if i is odd and $C_{b,n(i)}$ if i is even, and the vertical disks $a_{\pm} \times I$ or $b_{\pm} \times I$ in the $(k+1)$th block of M. These surfaces fit together to make a properly embedded connected surface R which is disjoint from L. R is orientable if k is even and non-orientable if k is odd.

For an example of how these twisted saddles fit together to give a properly embedded connected surface, consider

$$T \times I/_{\alpha^2} T \times I.$$

View this as two blocks of a once-punctured torus bundle joined together by the homeomorphism α^2. Let the first block contain a copy of $C_{a,+2}$ (as in Figure 4.1 without L) and the second block contain a copy of $C_{b,+2}$. To attach, we 'push' the homeomorphism α^2 through the second block as in Figure 4.2. We

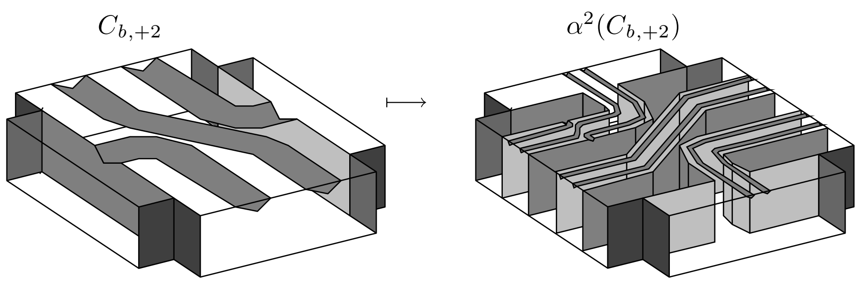


Figure 4.2: 'Pushing' α^2 through $C_{b,+2}$

may now join the blocks together to get Figure 4.3.

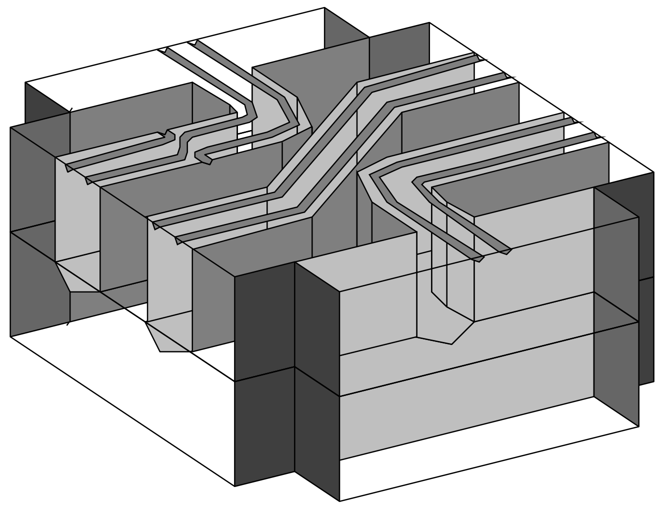


Figure 4.3: The twisted saddles $C_{a,+2}$ and $C_{b,+2}$ joined in $T \times I/_{\alpha^2} T \times I$

Definition 4.1.1. As in §2.5 of [6], we define the *twisted surface* $C(J; n(k), \ldots, n(1))$ to be R if k is even and $\partial N(R)$ if k is odd.

Note that:

- the surface $C(0; n(k), \ldots, n(1))$ has genus $\frac{1}{2}k - 1$ and four boundary components each of which intersects the fiber exactly once,
- the surface $C(2; n(k), \ldots, n(1))$ has genus $\frac{1}{2}k$ and two boundary components each of which intersects the fiber exactly twice, and
- the surfaces $C(-1; n(k), \ldots, n(1))$ and $C(+1; n(k), \ldots, n(1))$ both have genus k and two boundary components each of which intersects the fiber exactly four times.

Definition 4.1.2. Let S and S' be properly embedded connected orientable surfaces in once-punctured torus bundles M and M' respectively that are disjoint from essential level knots L and L' respectively. Then we say S and S' are of the *same type* if there is a bundle equivalence from M to M' that maps S to S' and L to L'. (C.f. §2.6 of [6].)

Remark 4.1.1. If the bundle M contains a surface of type $C(J; n(k), \ldots, n(1))$, then M has the characteristic class

$$[P^J B^{n(k)} \ldots B^{n(2)} A^{n(1)}].$$

4.1.2 Essential Twisted Surfaces

Proposition 4.1.1. *(c.f. Proposition 2.5.1 [6]) Let J, $n(1), n(2), \ldots, n(k)$, L, and M be as above. The surface $C(J; n(k), \ldots, n(2), n(1))$ is essential in $M - N(L)$ if and only if $|n(i)| \geq 2$ for $i = 2, \ldots, k$.*

Proof. We cite the proof of Proposition 2.5.1 [6] and show only the parts where we must diverge. We begin by reestablishing notation.

Let $\widetilde{M}$ be the cyclic cover, corresponding to the fiber, of the bundle

$$M = T \times I/_{\alpha^{n(1)}} T \times I/_{\beta^{n(2)}} \ldots T \times I/_{\varphi^J}.$$

Let $\widetilde{L}$ be the inverse image of L under the covering projection and

$$S \subset \widetilde{M} - N(\widetilde{L}) \subset \widetilde{M}$$

be a component of the inverse image of

$$C(J; n(k), \ldots, n(1)) \subset M - N(L) \subset M$$

under the covering projection. As noted in [6], it suffices to show that S is incompressible in $\widetilde{M} - N(\widetilde{L})$. Furthermore $\widetilde{M}$ is divided into blocks which are inverse images of the blocks of M, and each block in $\widetilde{M}$ meets S in one disk as in Fig. 4 of [6]. Let F be the union of the fibers along which the blocks in $\widetilde{M}$ are joined.

We want to consider the family of all compressing and boundary compressing disks for S in $\widetilde{M} - N(\widetilde{L})$. This is equivalent to considering the family of compressing disks for S in $\widetilde{M}$ that are disjoint from $\widetilde{L}$. We first, however, consider the family of all compressing and boundary compressing disks for S in $\widetilde{M}$ (regardless of $\widetilde{L}$). By Proposition 2.5.1 of [6], this family is non-empty if and only if $|n(i)| < 2$ for some $i = 1, \ldots, k$.

From the proof of Proposition 2.5.1, such a minimal disk is contained in the solid torus formed by cutting two adjacent blocks of $\widetilde{M}$ along S and joining two of the resulting solid torus components along the annulus of their common intersection with F. If the two blocks are preimages of the $(i-1)$th (modulo k) and ith blocks of M under the covering projection, then $|n(i)| < 2$ if and only if such a disk exists in this solid torus. The disk is isotopic to a meridional disk of the solid torus and non-trivially intersects the core of the solid torus. In the case that $|n(1)| < 2$, a component of $\widetilde{L}$ is contained in this solid torus (since it is isotopic to the core of the gluing annulus) and is isotopic to its core. Thus it will non-trivially intersect the disk.

$\square$

As stated in Remark 2.5.2 of [6], a stronger statement than that of the above proposition is true. We will need this stronger statement and hence prove it here.

Theorem 4.1.2. *(c.f. Remark 2.5.2 [6]) Let $J, n(1), n(2), \ldots, n(k)$, L and M be as above. Let $\widehat{M}$ be the manifold constructed by attaching a solid torus to M so that the boundary curves of $C(J; n(k), \ldots, n(2), n(1))$ are identified to contractible curves in the solid torus. Let $\widehat{C}(J; n(k), \ldots, n(2), n(1))$ be the surface in $\widehat{M}$ obtained by attaching disks to the boundary curves of $C(J; n(k), \ldots, n(2), n(1))$. Then $\widehat{C}(J; n(k), \ldots, n(2), n(1))$ is essential in $\widehat{M} - N(L)$ if and only if $|n(i)| \geq 2$ for $i = 2, \ldots, k$.*

Proof. We will first prove this only in the case that $J = 0$ or 2. Afterwards we will address the case that $J = \pm 1$.

Let $C = C(J; n(k), \ldots, n(2), n(1))$ and $\widehat{C} = \widehat{C}(J; n(k), \ldots, n(2), n(1))$. If $J = 0$ or 2, then each block meets C in a single disk as in Fig. 4 of [6]. If $J = \pm 1$, then each block meets C in two disks which are a parallel, each of which individually appears as in Fig 4. of [6]. Let F be the union of the fibers along which the blocks of M are joined.

Let K be the core of the surgered torus so that $\widehat{M} - N(K) = M$. Consider the family of all compressing disks for $\widehat{C}$ in $\widehat{M} - N(L)$. If this family is non-empty then there exists a member D for which $(D \cap K, D \cap F)$ is minimized lexicographically. We may assume D transversely intersects K.

Due to Proposition 4.1.1, $D \cap K \neq \emptyset$. Let $P = D - N(K) \subset M - N(L)$ be the punctured disk which has one boundary component, say $\partial P_0 = \partial D$, on C and all other boundary components on $\partial N(K)$ parallel to ∂C. Note that P is incompressible in $M - N(L)$ due to the minimality assumption on D.

First we observe that there cannot be any simple closed curve components of $P \cap F$ contractible on F. If there is, then there is one, say σ', which is innermost on F. Thus σ' bounds a disk $\Delta \subset F$. Surgering P along Δ yields two surfaces one of which, say P', has ∂D as a boundary component. P' then completes to a compressing disk D' for $\widehat{C}$ in $\widehat{M} - N(L)$ with $\partial D' = \partial D$. Furthermore $|D' \cap K| \leq |D \cap K|$ and $|D' \cap F| < |D \cap F|$. This contradicts the minimality assumption on D.

Part I. $J = 0$ or 2

Let X be a block of M_K with $\partial P_0 \cap X \neq \emptyset$. Cutting X along C yields two solid tori as shown in Fig. 6 of [6]. Consider one of the solid tori which has non-empty intersection with ∂P_0. As before (at the beginning of this proof) its boundary meets C in a disk, ∂M in two disks (labelled B), and F in a disk, F_d, and an annulus, F_a. Note that ∂P_0 need not intersect F_d.

Suppose that σ is an arc component of $P \cap F_d$. There are four cases for such an arc that we need to consider.

Case 1. Both end points of σ are on the same component of $B \cap F_d$. Here σ is an arc with both end points in $\partial P - \partial D \subset \partial M$. Among all the components of $P \cap F_d$ that have both end points on the same component of $B \cap F_d$, choose one, say σ', that is outermost on F_d. Thus there is a disk $\Delta \subset F_d$ such that $\partial \Delta$ consists of two arcs, σ' and δ' where $\delta' \subset R \cap F_d$ and $\Delta \cap P = \sigma'$. δ' is an arc on ∂M connecting two distinct components of ∂P on ∂M. Let Q be the annulus of ∂M connecting these two components that contains δ'. Surgering P along $\Delta \cup Q$ yields a planar surface P' such that $\partial D \subset \partial P'$. Then in M_L P' completes to a compressing disk D' for which $|D' \cap K| < |D \cap K|$ contradicting the minimality assumption on D.

Case 2. Both end points of σ are on the same component of $C \cap F_d$.

Here σ is an arc with both end points in $\partial D \subset C$. Among all the components of $P \cap F_d$ that have both end points on the same component of $C \cap F_d$, choose one, say σ', that is outermost on F_d. Thus there is a disk $\Delta \subset F_d$ such that $\partial\Delta$ consists of two arcs, σ' and δ' where $\delta' \subset R \cap F_d$ and $\Delta \cap P = \sigma'$. Surgering P along Δ yields two planar surfaces each with one boundary component on C and the rest on ∂M. Both of these planar surfaces complete to disks, say D' and D'' in M_L with boundary in $\widehat{C}$. Both of these disks have fewer intersections with F and no more intersections with K than D. At least one of the boundaries of these disks, say D', must be essential in $\widehat{C}$. Thus D' is a compressing disk for $\widehat{C}$ contradicting the minimality assumption on D.

Case 3. One end point of σ is contained in B and one is contained in C. On P, σ must connect ∂D with another boundary component of P. Since neither Case 1 nor Case 2 occur, we may choose among all the components of $P \cap F_d$ with one end point in B and one end point in C an arc, say σ', that is outermost on F_d. Thus there is a disk $\Delta \subset F_d$ such that $\partial\Delta$ is the union of three consecutive arcs σ', $r' \subset C$, and $\delta' \subset B$, and $\Delta \cap P = \sigma'$. δ' connects a component of ∂P and ∂C on ∂M. Let Q be the annulus of ∂M between these two components that contains δ'. Surger P along $\Delta \cup Q$ to yield a surface P'. In M, P' completes to a disk D' with $\partial D'$ isotopic to ∂D in $\widehat{C}$. Thus D' is a compressing disk for $\widehat{C}$. Furthermore, $|D' \cap K| < |D \cap K|$ contradicting the minimality assumption on D.

Case 4. One end point of σ is contained in one component of $C \cap F_d$ and the other end point is contained in the other component of $C \cap F_d$. On P, σ has both end points on $\partial D \subset C$. In F_d, the arc σ is parallel in F_d into an arc component of $F_d \cap B$. Since none of the Cases 1, 2, or 3 occur, among all the components of $P \cap F_d$ with end points in $C \cap F_d$ there is one, say σ', that is outermost on F_d. Thus there is a disk $\Delta \subset F_d$ such that $\partial\Delta$ is the union of four consecutive arcs σ', $r' \subset C$, $\delta' \subset B$, and $r'' \subset C$, and $\Delta \cap P = \sigma'$. δ' connects two components of ∂C on ∂M. Let Q be the annulus of ∂M between these two components that contains δ'. Surger P along $\Delta \cup Q$ to yield a surface P'. In M P' completes to a disk D' with $\partial D'$ isotopic to ∂D in $\widehat{C}$. Thus D' is a compressing disk for $\widehat{C}$. Furthermore, $|D' \cap K| < |D \cap K|$ contradicting the minimality assumption on D.

Note that there is fifth possible arc type for σ: One end point of σ is contained in one component of $B \cap F_d$ and the other end point is contained in the other component of $B \cap F_d$. We need not address this case for our purposes.

Let X' denote the block in M that meets the block X along the component of F containing F_a. X' splits into two solid tori as X did, and the same arguments of Cases 1, 2, 3, and 4 for X apply to X'. Joining X and X' along the annulus F_a forms a solid torus V which meets C in an annulus that wraps around the solid torus $|n(i)|$ times for some i.

Note that as a consequence of Cases 2, 3, and 4, $F_d \cap \partial P_0 = \emptyset$. Therefore ∂P_0 is contained in the annulus $C \cap V$ and is thus isotopic to the core of this annulus.

Consider the component Q of $P \cap V$ that contains ∂P_0. Since P is incompressible in $M - N(L)$, then Q must be incompressible in $V - N(L)$ which is either a solid torus or a torus cross interval depending on whether

L is contained in V. Since the only incompressible surfaces in a solid torus are meridional disks and boundary parallel annuli, either Q is one of these if $L \not\subset V$ or Q is a boundary parallel annulus if $L \subset V$.

Case A. Assume Q is a boundary parallel annulus. Since one component of ∂Q is ∂P_0 the other component does not intersect C. Let σ be an arc of $Q \cap F_a$ and Δ be the disk in F_a with $\partial \Delta = \sigma \cup (\Delta \cap \partial V)$. Thus σ has one end point in C and one end point in B. Hence σ is an arc like in Case 3. Since Cases 1, 2, and 3 may apply to arcs of $P \cap \Delta$ in Δ, the existence of σ yields a contradiction.

Case B. Assume Q is a meridional disk. Such a disk can only occur if $n(i) = 0$ which cannot be the case unless $i = 1$. However, if $i = 1$, then $L \subset V$ and then $L \cap Q \neq \emptyset$ which is a contradiction.

Part II. $J = \pm 1$

Again, let X be a block of M_K with $\partial P_0 \cap X \neq \emptyset$. Cutting X along C yields two solid tori as shown in Fig. 6 of [6] as well as one product disk ($\cong D^2 \times I$) that gives the parallelism between the two disk components of $X \cap C$.

The arguments of Part I apply to these solid torus components. Therefore we only need consider the case that the product disk has non-empty intersection with ∂P_0. The boundary of the product disk meets C in two disks, ∂M in four disks (labelled B), and F in four disks.

Each disk component of the intersection of the product disk and F is a rectangle with two edges on C and two edges on B. Consider one of these components which has non-empty intersection with ∂P_0 and call it F_d.

Cases 1, 2, 3, and 4 of Part I all then apply to the arcs of $P \cap F_d$. Therefore $F_d \cap \partial P_0 = \emptyset$. Thus ∂P_0 must be contained in this product disk. But this cannot occur as then ∂P_0 would be contained in a disk of C and hence bound a disk in C.

$\square$

Definition 4.1.3. We may say that a surface S_0 in $\widehat{M} - N(L)$ has the *same type* as $\widehat{C}(J; n(k), \ldots, n(2), n(1))$ if it is isotopic to $\widehat{C}(J; n(k), \ldots, n(2), n(1))$ in $\widehat{M} - N(L)$.

Remark 4.1.2. Continuing with the notation above, if $\widehat{M} \cong S^3$ and $|n(i)| \geq 2$ for $i = 2, \ldots, k$, then L is a knot in S^3 and $\widehat{C}(J; n(k), \ldots, n(2), n(1))$ is a closed essential surface in the complement of L, $X_L = S^3 - N(L)$.

4.1.3 Classification of Essential Twisted Surfaces

A given once-punctured torus bundle with an essential level knot may contain several surfaces essential in the complement of the level knot which are have the type of a twisted surface. We may determine when two are in the same isotopy class. This is effectively done in §4.1 of [6].

Recall that if M is a fiber bundle, then an isotopy of M which is a bundle equivalence at each time is called a *bundle isotopy.*

Proposition 4.1.3. *Let M be a once-punctured torus bundle containing surfaces S and S' of types $C(J; n(k), \ldots, n(1))$ and $C(J'; m(k'), \ldots, m(1))$, respectively, that are disjoint from an essential level curve L. Then S and S' are isotopic in the complement of L if and only if $k = k'$, $J = J'$, and*

$(m(k), \ldots, m(1)) = (n(k), \ldots, n(1))$. *Moreover there is a bundle isotopy taking* (S, L) *to* (S', L).

Proof. This follows directly from the proof of Proposition 4.1.3 of [6]. Note that because of L, we get that $(m(k), \ldots, m(1))$ and $(n(k), \ldots, n(1))$ are related by equality rather than cyclic permutation. $\square$

We now relate essential twisted surfaces in a based once-punctured torus bundle to certain expressions of the bundle's characteristic class (c.f. Remark 4.1.1).

Proposition 4.1.4. *Let* L *be an essential level curve in a once-punctured torus bundle* M. *Assume* $M = T \times I/_\eta$ *such that* L *is contained in* $T \times \{0\}$. *Assume* η *is identified with* $H \in \mathrm{SL}_2(\mathbb{Z})$. *Then for each element* $W \in \mathrm{SL}_2(\mathbb{Z})$ *such that* $W[a] = [L]$: *if* $W^{-1}HW$ *may be expressed as a special form*

$$P^J A^{n(k)} \ldots B^{n(2)} A^{n(1)}, \text{ where } J \in \{+1, -1\} \text{ and } |n(i)| \geq 2 \text{ for } i = 2, \ldots, k$$

or

$$P^J B^{n(k)} \ldots B^{n(2)} A^{n(1)}, \text{ where } J \in \{0, 2\} \text{ and } |n(i)| \geq 2 \text{ for } i = 2, \ldots, k$$

then there is an essential surface in $M - N(L)$ *of type* $C(J; n(k), \ldots, n(1))$.

Proof. Let ω be the homeomorphism of T associated to W. Then ω naturally extends to a bundle equivalence

$$\omega : M' = T \times I/_{\omega^{-1} \circ \eta \circ \omega} \rightarrow M = T \times I/_\eta$$

such that $\omega(a) = L$. If $W^{-1}HW$ may be expressed as the special form $P^J \dots B^{n(2)}A^{n(1)}$ then

$$\omega^{-1} \circ \eta \circ \omega = \phi^J \dots \beta^{n(2)}\alpha^{n(1)}.$$

Thus

$$M' = T \times I/_{\omega^{-1}\circ\eta\circ\omega} = T \times I/_{\phi^J \dots \beta^{n(2)}\alpha^{n(1)}},$$

and so M' contains the twisted surface $C(J; n(k), \dots, n(2), n(1))$. Since $P^J \dots B^{n(2)}A^{n(1)}$ is a special form, $|n(i)| \geq 2$ for $i = 2, \dots, k$, and hence the surface $C(J; n(k), \dots, n(2), n(1))$ is essential in the complement of $a \times \{0\}$.

Therefore $\omega(C(J; n(k), \dots, n(2), n(1)))$ is an essential surface in

$$M - N(L) = \omega(M') - N(\omega(a))$$

of type $C(J; n(k), \dots, n(2), n(1))$. $\square$

Together, Propositions 4.1.3 and 4.1.4 give us

Corollary 4.1.5. *Let M, L, and H be as above. The essential surfaces in $M - N(L)$ with the type of a twisted surface are in one-to-one correspondence with special forms of conjugates $W^{-1}HW$ of H where $W \in \mathrm{SL}_2(\mathbb{Z})$ such that $W[a] = [L]$.*

4.2 Structure of Surfaces

This section closely follows §3 of [6]. Many of their arguments are easily modified to accommodate the presence of an essential level knot L in a once-punctured torus bundle M. Parts that do not involve L will be simply cited.

Let S be a properly embedded surface in M that is disjoint from L and essential in $M - N(L)$. We will say that S is in *general position* provided that

1. each component of ∂S is either contained in a fiber or is transverse to every fiber,

2. the projection $p : M \to S^1$ restricts to a Morse function on the interior of S having distinct critical values different from $p(L)$,

3. among all surfaces isotopic to S in $M - N(L)$ and satisfying (1) and (2), S has the minimal number of index 0 or 2 critical points.

We may assume that S has been moved by an isotopy with support outside of a neighborhood of L to be in general position. The level sets of $p|_S$ are the intersections of S with the fibers of M.

Let x be a critical point. We use the following terms as defined in section 3 of [6]: *level arcs*, *level sets*, *critical neighborhood* of x, and *upper* and *lower level sets of* x. We let X be the critical neighborhood of x. We then have the following lemmas:

Lemma 4.2.1. (3.1.1 of [6]) *Each level arc of S is essential in the fiber containing it.*

Lemma 4.2.2. (3.2.1 of [6]) *Either S meets every non-critical fiber only in arcs or S meets every non-critical fiber only in simple closed curves.*

Lemma 4.2.3. (3.2.2 of [6]) *If both the upper and lower level sets of X contain an essential closed curve, then these curves are isotopic.*

Lemma 4.2.4. (3.2.3 of [6]) *If the lower level set contains two arcs then*

1. *they are parallel, and*

2. *the upper level set is obtained by a band sum across the annulus component of the complement.*

Proof. All four lemmas follow almost exactly as in [6].

One needs for the proof of Lemma 4.2.2 that $M - N(L)$ is irreducible. This follows from the irreducibility of M and that L is non-trivial in its fiber.

Since critical points of S occur away from L, arguments involving critical neighborhoods are unchanged. $\square$

For the proof of Theorem 3.3.1 in [6], the authors employ a lemma due to Haken [9]. For the upcoming proof of Theorem 4.2.6, we must alter the lemma to accommodate the presence of L.

Lemma 4.2.5. *Let T be a punctured torus, and let L be an essential, non-boundary parallel simple closed curve in the fiber $T \times \{\frac{1}{2}\}$ of $T \times I$. Let R be a properly embedded, connected, incompressible surface in $T \times I - N(L)$ such that each component of ∂R is contained in either $\partial T \times I$, $T \times \{0\}$, or $T \times \{1\}$ and, in the latter two cases, parallel in its fiber to $\partial T \times I$. Then R is either an annulus, a once-punctured torus, or a torus parallel to $\partial N(L)$.*

Proof. R is disjoint from L in $T \times I$. If R is incompressible in $T \times I$, then it follows from [9] that R is either an annulus or a once-punctured torus. So assume R is compressible in $T \times I$, and note that any compressing disk must then intersect L.

We may assume that $L = a \times \{\frac{1}{2}\}$. Then let A_a be the annulus $a \times I$, and let A_b be the annulus $b \times I$. Note that L is the core of A_a and intersects A_b once.

By standard arguments, we may assume that R has been isotoped rel-∂ to minimize both $|A_a \cap R|$ and $|A_b \cap R|$. It follows that $A_a \cap R$ is a collection of simple closed curves on A_a that are parallel to the core of A_a (and hence L), and that $A_b \cap R$ is a collection of simple closed curves on A_b that either bound disks in A_b that intersect L or are parallel to the core of A_b.

Case 1. Assume there exists a curve in $A_b \cap R$ that bounds a disk in A_b. Among the curves of $A_b \cap R$ that bound disks in A_b, let c be the innermost. Let D_c be the disk bounded by c. By minimality assumptions, we may assume $|c \cap A_a| = 2$. Thus there exists a curve in A_a, say c', such that $c \cap c' \neq \emptyset$. Note that c' is innermost among $A_a \cap R$ in the sense that if $A_{c'}$ is the annulus between L and c', then $A_{c'} \cap R = c'$. Consider $\delta = \partial N(c \cup c') \cap R$ and the subdisk Δ of $\partial N(D_c \cup A_{c'})$ that it bounds. Note that since Δ is disjoint from L, δ must bound a disk, say Δ', in R. Furthermore, note that $\Delta' \cap N(c \cup c') = \delta$. Thus $R = \Delta' \cup N(c \cup c')$ which is a torus. This torus is parallel to $\partial N(L)$.

Case 2. Assume now that there are no curves of $A_b \cap R$ that bound disks in A_b. Thus all curves of $A_b \cap R$ are parallel to the core of A_b. Among the compressing disks for R, let D be a compressing disk that $|D \cap A_a|$ is minimal. Thus $D \cap A_a$ contains no simple closed curves. Note that $D \cap A_a \neq \emptyset$ since $D \cap L \neq \emptyset$. Consider the arcs of $D \cap A_a$. Due to the minimality assumptions on $D \cap A_a$ and $R \cap A_a$, every arc of $D \cap A_a$ crosses L.

Let d be an outermost arc on D of $D \cap A_a$, and let D' be the (outermost) disk cut off by d. Let d' be the arc of ∂D so that $\partial D' = d' \cup d$. The endpoints of d lie on curves of $A_a \cap R$, say c and c', that are each adjacent to L. Let $A_{cc'}$ be the annulus in A_a between c and c'. Consider $\delta = \partial N(c \cup c' \cup d') \cap R$ and the subdisk Δ of $\partial N(A_{cc'} \cup D')$ that it bounds. Since $\Delta \cap L = \emptyset$, δ must bound a disk, say Δ', in R. Therefore we have the annulus $\Delta' \cup N(c \cup c' \cup d')$ which intersects A_b in an arc r of some component c'' of $A_b \cap R$. Since r connects $c \cap A_b$ to $c' \cap A_b$, c'' intersects $A_a \cap A_b$ at least twice. However, since the curves of $A_b \cap R$ are parallel to the core, by the minimality assumptions on $|A_a \cap R|$, $|c'' \cap (A_a \cap A_b)| = 1$. Thus we have a contradiction. □

Theorem 4.2.6. *Let M be a once-punctured torus bundle such that the characteristic class of M does not have trace 2. Let L be an essential level curve in M. If $(S, \partial S) \subset (M - N(L), \partial M)$ is an essential connected surface in $M - N(L)$ such that ∂S is transverse to the induced fibration on ∂M, then S is isotopic to a surface of type $C(J; n(k), \ldots, n(1))$.*

Remark 4.2.1. Theorem 4.2.6 is true in greater generality. One may prove a theorem analogous to Theorem 3.3.1 of [6] for surfaces in the complement of an essential level knot such as L. Perhaps this may be generalized further by considering fibrations (or even foliations) other than once punctured torus bundles.

Proof. This proof largely follows the proof of Theorem 3.3.1 in [6]. We will sketch only the parts of their proof that are needed for the parts that need alterations. We assume S has been isotoped into general position. By Lemma 4.2.2 (Lemma 3.2.1 of [6]) S meets every non-critical fiber only in arcs or S meets every non-critical fiber in simple closed curves.

Case 1. S meets every non-critical fiber only in simple closed curves.

In the first part of this case we assume we have a non-critical fiber F (not containing L) with S intersecting F in only inessential (or boundary parallel) curves. By a further isotopy, we assume that the number of such curves is minimal. Thus either $S \cap F = \emptyset$ or every component of $S \cap F$ is parallel into ∂F.

Splitting M along F yields the product $T \times I$ and induces a splitting of S. Let S' be the surface resulting from splitting S along F. S' is incompressible in $T \times I - N(L)$, and it follows that every component of S' is just like R in Lemma 4.2.5. Thus every component of S' is either an annulus, a once-punctured torus, or a torus parallel to $\partial N(L)$. A component of S' cannot be a torus parallel to $\partial N(L)$ since then $S = S'$ and contradicts the assumption that S is essential.

By considering the placement of the boundary of such an annulus or once punctured torus, the minimality condition imposed on $S \cap F$ forces two things. Either $S \cap F$ has one component and S' is an annulus parallel into $\partial T \times I$ or $S \cap F = \emptyset$. Hence S is either a torus parallel to ∂M or S is a once punctured torus isotopic to a fiber. Neither of these may occur as S is not essential in the former and ∂S is not transverse to the fibration in the latter.

In the second part of this case we assume every non-critical fiber F contains an essential simple closed curve component of $S \cap F$. From here, we follow [6] exactly to the conclusion that the characteristic class of M fixes the isotopy class of an essential curve. Hence the characteristic class of M has trace 2, contrary to assumption.

Case 2. S meets every non-critical fiber in essential arcs. The fiber containing L must have exactly one non-empty family of parallel level arcs. This allows us to skip much of Case 2 of Theorem 3.3.1 of [6].

Let F be a fiber in a neighborhood of the fiber containing L. Thus S meets F in one family of parallel arcs. If S has no critical points, then the characteristic class of M must fix the isotopy class of an essential simple closed curve and thus have trace 2. By assumption, this is not the case. Thus S must contain a critical point. The proof completes just as the last paragraph of Theorem 3.3.1 of [6]. It follows that S is a surface of type $C(J; n(k), \ldots, n(1))$.

□

4.3 Framing

4.3.1 Review

Let us recall the discussion about framing in §6.2 of [6]. We will use the definitions and notation (and its abuse) established there. The following is paraphrased from that section:

Fix a base point $x \in \partial T$ and let a and b be elements of $\pi_1(T, x)$ analogous to the a and b in Figure 2.5. Let $\mathrm{Stab}([a, b])$ be the subgroup of the automorphisms of $\pi_1(T, x)$ that stabilizes $[a, b]$ and therefore act as identity on ∂T (i.e. up to isotopy fixing x). Elements $\gamma \in \mathrm{Stab}([a, b])$ correspond uniquely to homeomorphisms g of (T, x) such that $g_* = \gamma$ and to simple closed curves t_γ on the boundary of $M = T \times I/_g$ that are transverse to the fibration such that $t_\gamma a t_\gamma^{-1} = \gamma(a)$ and $t_\gamma b t_\gamma^{-1} = \gamma(b)$. Together with the boundary of a fiber (which is analogous to the standard longitude of a knot in S^3), the curve t_γ

defines a basis, or framing, for $H_1(\partial M)$. Therefore:

Definition 4.3.1. An element γ of $\mathrm{Stab}([a,b])$ is a *framing* for a once-punctured torus bundle $M = T \times I/_g$ where $g_* = \gamma$.

If g' is also a homeomorphism of (T, x) that is isotopic (not necessarily fixing x) to g, then $M' = T \times I/_{g'}$ is bundle equivalent to $M = T \times I/_g$ by say h.

The corresponding framings $\gamma = g_*$ and $\gamma' = g'_*$ differ by a conjugation by $[a,b]^j$ for some $j \in \mathbb{Z}$, i.e. $\gamma^{-1} h_* \gamma' h_*^{-1}$ is a conjugation by $[a,b]^j$. This j describes the difference in the number of times t_γ and $t_{\gamma'}$ spin around the boundary of the fiber.

Definition 4.3.2. This number $j \in \mathbb{Z}$ is called the *transition index* between γ and γ'.

Given a framing for a once-punctured torus bundle, we may describe the boundary of an essential surface of type $C(J; n(k), \ldots, n(1))$ in terms of this framing. More precisely, we may associate a framing to the special form corresponding to the essential twisted surface and understand the boundary curves of the twisted surface in terms of this framing.

The *standard framings* defined in [6] are as follows:

$$\alpha\colon \begin{cases} a \to a \\ b \to ba^{-1} \end{cases} \qquad \text{is standard for } A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix},$$
$$\beta\colon \begin{cases} a \to ab \\ b \to b \end{cases} \qquad \text{is standard for } B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$
$$\phi\colon \begin{cases} a \to aba^{-1} \\ b \to a^{-1} \end{cases} \qquad \text{is standard for } P = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$
$$\psi\colon \begin{cases} a \to ab^{-1} \\ b \to bab^{-1} \end{cases} \qquad \text{is standard for } Q = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}.$$

Note that α, β, ϕ, and ψ are being used duplicitously for elements of $\mathrm{Stab}([a,b])$ and homeomorphisms of T. Using these standard framings, we may obtain standard framings for elements of $\mathrm{SL}_2(\mathbb{Z})$ corresponding to special forms.

4.3.2 Stab([a, b])

One may readily observe that $\mathrm{Stab}([a,b])$ is generated by the standard framings α and β together with the framing

$$\delta\colon x \to [a,b]x[a,b]^{-1}$$

induced by a Dehn twist along a simple curve parallel to ∂T.

We may find by direct computations that $\alpha = \psi\phi$, $\beta = \phi\psi$, and $\delta = \phi^4$. Hence $\mathrm{Stab}([a,b])$ is generated by the standard framings ϕ and ψ. By a further computation we obtain the relationship $\phi^2\psi^3 = 1$. One may check that any other relation between ϕ and ψ is a consequence of this one. Therefore we have the presentation

$$\mathrm{Stab}([a,b]) = \langle \phi, \psi | \phi^2\psi^3 = 1\rangle.$$

Note that ϕ^2 and ψ^3 are both in the center of $\mathrm{Stab}([a, b])$. Furthermore, if j is the transition index between two framings ξ and ζ, then

$$\xi^{-1}\mu\zeta\mu^{-1} = \phi^{4j}$$

for some framing μ.

4.3.3 Exponent sums

Definition 4.3.3. Let w be a word on the set of letters Ω, say $w = \prod_{i=1}^{k} x_i^{e_i}$ where $x_i \in \Omega$ and $e_i \in \mathbb{Z}$. Define the *exponent sum* of the word w to be $E(w) = \sum_{i=1}^{k} e_i$. Define the exponent sum of the letter x for the word w to be $E_x(w) = \sum_{x_i = x} e_i$.

Lemma 4.3.1. *Let ζ be a framing for $Z \in \mathrm{SL}_2(\mathbb{Z})$, and let ξ be a framing for X conjugate to Z. Let ω be a word for $\zeta^{-1}\xi$ written in ϕ and ψ. Then*

$$(E_\phi(\omega), E_\psi(\omega)) = (4j, 0) \mod (2, 3)$$

where j is the transition index between ζ and ξ.

Proof. First we will show that $(E_\phi(\omega), E_\psi(\omega)) \mod (2, 3)$ is independent of choice of the spelling of the word ω.

Let ω and ω' be two words in ϕ and ψ representing an element of $\mathrm{Stab}([a, b])$. The passage between the two words occurs by the following moves:

(i) insertion or deletion of an adjacent pair of a letter and its inverse and

(ii) application of the relation $\phi^2\psi^3 = 1$.

Assume ω and ω' differ by just one of these moves. Let us examine the effects upon the exponent sums. Since (i) only introduces or removes cancelling pairs, it does not change the exponent sums at all. For (ii), let us assume ω' is obtained from ω by replacing $\phi^2\psi^3$ with 1. Then

$$(E_\phi(\omega'), E_\psi(\omega')) = (E_\phi(\omega) - 2, E_\psi(\omega) - 3) = (E_\phi(\omega), E_\psi(\omega)) - (2,3)$$

Thus

$$(E_\phi(\omega'), E_\psi(\omega')) = (E_\phi(\omega), E_\psi(\omega)) \mod (2,3)$$

Let j be the transition index between ζ and ξ. Thus $\zeta^{-1}\mu\xi\mu^{-1} = \phi^{4j}$ for some framing μ for $U \in \mathrm{SL}_2(\mathbb{Z})$ such that $Z^{-1}UXU^{-1} = I$. Let ω_ζ, ω_ξ, and ω_μ be words in ϕ and ψ for ζ, ξ, and μ respectively. Thus we may write

$$\omega_\zeta^{-1}\omega_\mu\omega_\xi\omega_\mu^{-1} = \phi^{4j}.$$

Therefore since $\zeta^{-1}\mu\xi\mu^{-1} = \phi^{4j}$, the words $\omega_\zeta^{-1}\omega_\mu\omega_\xi\omega_\mu^{-1}$ and ϕ^{4j} represent the same element of $\mathrm{Stab}([a,b])$. Hence it follows that

$$\begin{aligned}(E_\phi(\omega_\zeta^{-1}\omega_\mu\omega_\xi\omega_\mu^{-1}), E_\psi(\omega_\zeta^{-1}\omega_\mu\omega_\xi\omega_\mu^{-1})) &= (E_\phi(\phi^{4j}), E_\psi(\phi^{4j})) &&\mod (2,3)\\ &= (4j, 0) &&\mod (2,3).\end{aligned}$$

Moreover, notice that $E_\phi(\omega_\mu) = -E_\phi(\omega_\mu^{-1})$ and $E_\psi(\omega_\mu) = -E_\psi(\omega_\mu^{-1})$, and thus

$$(E_\phi(\omega_\zeta^{-1}\omega_\mu\omega_\xi\omega_\mu^{-1}), E_\psi(\omega_\zeta^{-1}\omega_\mu\omega_\xi\omega_\mu^{-1})) = (E_\phi(\omega_\zeta^{-1}\omega_\xi), E_\psi(\omega_\zeta^{-1}\omega_\xi)).$$

Since $\omega_\zeta^{-1}\omega_\xi$ is a word for $\zeta^{-1}\xi$, the conclusion of the lemma follows. □

Assume $P^J \dots B^{n(2)}A^{n(1)}$ is a special form for an element X of $\mathrm{SL}_2(\mathbb{Z})$. This corresponds to an essential twisted surface in the complement of an essential level knot in a once-punctured torus bundle M with characteristic class

$[X]$. Consider the standard framing $\xi = \phi^J \ldots \beta^{n(2)}\alpha^{n(1)}$ corresponding to the specific form for X. Assume a framing ζ has been chosen for M, so that $\zeta \to Z \in \mathrm{SL}_2(\mathbb{Z})$. Note that $Z^{-1}UXU^{-1} = I$ for some $U \in \mathrm{SL}_2(\mathbb{Z})$ whereas $\zeta^{-1}\mu\xi\mu^{-1} = \phi^{4j}$ where μ is any framing for U and $j \in \mathbb{Z}$ is the transition index.

Corollary 4.3.2. *Continuing with the above notation, let $M \cong S^3 - N(K)$ for some genus one fibered knot K. Assume Z may be written as a word W_Z in A and B so that the corresponding standard framing is meridional. The boundary components of $C(J; n(k), \ldots, n(2), n(1))$ are meridional if and only if $J = 0$ and the exponent sum $E(W_Z)$ equals $\sum_{i=1}^{k} n(i)$.*

Proof. A boundary component of $C(J; n(k), \ldots, n(2), n(1))$ intersects a fiber of M only once if and only if $J = 0$. The transition index between ξ and ζ measures how many times the boundary component wraps longitudinally. Thus the boundary components of $C(J; n(k), \ldots, n(2), n(1))$ are meridional if and only if $J = 0$ and the transition index between ξ and ζ is zero.

Note that the standard framings for A and B are $\alpha = \psi\phi$ and $\beta = \phi\psi$ respectively. Thus E_ϕ and E_ψ are equal on α and β.

Also note that for $\omega \in \mathrm{Stab}([a, b])$,

$$(E_\phi(\omega), E_\psi(\omega)) = -(E_\phi(\omega^{-1}), E_\psi(\omega^{-1})).$$

Thus given a word ω written in α and β,

$$(E_\phi(\omega), E_\psi(\omega)) = E(\omega)(1, 1) \mod (2, 3).$$

Let ω_ζ be the standard framing for W_Z written as a word in α and β. (Note that ω_ζ^{-1} is a word for ζ^{-1}.) Then since $\xi = \phi^J \dots \beta^{n(2)}\alpha^{n(1)}$ and $J = 0$, let ω_ξ be the word $\beta^{n(k)} \dots \beta^{n(2)}\alpha^{n(1)}$.

Thus we have $\zeta^{-1}\xi$ written as the word $\omega_\zeta^{-1}\omega_\xi$ in α and β. Therefore

$$\begin{aligned}(E_\phi(\omega_\zeta^{-1}\omega_\xi), E_\psi(\omega_\zeta^{-1}\omega_\xi)) &= E(\omega_\zeta^{-1}\omega_\xi)(1,1) \\ &= (E(\omega_\zeta^{-1}\omega_\xi), E(\omega_\zeta^{-1}\omega_\xi)) \mod (2,3).\end{aligned}$$

Since we need the transition index to be zero, we need

$$(E(\omega_\zeta^{-1}\omega_\xi), E(\omega_\zeta^{-1}\omega_\xi)) = (0,0) \mod (2,3).$$

In other words, it must be that

$$(E(\omega_\zeta^{-1}\omega_\xi), E(\omega_\zeta^{-1}\omega_\xi)) - N(2,3) = (0,0)$$

for some $N \in \mathbb{Z}$. This can only happen in the case that $E(\omega_\zeta^{-1}\omega_\xi) = 0$ and $N = 0$. Therefore $E(\omega_\zeta) = E(\omega_\xi)$.

Since $E(\omega_\zeta) = E(W_Z)$ and $E(\omega_\xi) = \sum_{i=1}^{k} n(i)$, the conclusion of the corollary follows. $\square$

4.4 Algorithms

We now give an algorithm to list all the essential surfaces of type $C(J; n(k), \dots, n(1))$ in the complement of a level knot in a once-punctured torus bundle.

Algorithm 4.4.1. *Let M be a once-punctured torus bundle given by an element $H \in \mathrm{SL}_2(\mathbb{Z})$. Let L be an essential simple closed curve on the fiber*

$T \times \{0\}$ of M given by the ordered pair $(x, y) \in \mathbb{Z}^2$. The following steps give a procedure to list all essential surfaces of type $C(J; n(k), \ldots, n(1))$ in $M - N(L)$.

Step 1. *Choose a change of basis matrix $W \in \mathrm{SL}_2(\mathbb{Z})$ so that $W \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$.*

Step 2. *Let $X(N) = (WA^N)^{-1} H (WA^N)$.*

Step 3. *List all N such that $X(N) = \begin{pmatrix} p(N) & r(N) \\ q(N) & s(N) \end{pmatrix}$ has*

$$|p(N)| < |q(N)|$$

Step 4. *For each N listed in Step 3, obtain the minimal continued fraction expansions $[a_1, a_2, \ldots, a_{k-1}]$ for $p(N)/q(N)$ such that $[a_1, a_2, \ldots, a_{k-1}, a_k]$ is a continued fraction expansion for $r(N)/s(N)$.*

If k is odd, then for either $J = +1$ or $J = -1$,

$$P^J A^{a_1} B^{a_2} \ldots A^{a_{k-2}} B^{a_{k-1}} A^{a_k} = X(N).$$

If k is even, then for either $J = 0$ or $J = 2$,

$$P^J B^{a_1} A^{a_2} \ldots A^{a_{k-2}} B^{a_{k-1}} A^{a_k} = X(N).$$

Step 5. *For each N in Step 3, list the expressions for $X(N)$ obtained in Step 4.*

Step 6. *For every expression $P^J C^{n(k)} \ldots B^{n(2)} A^{n(1)}$ listed in Step 5 (where $C = A$ or B depending on the parity of k), there is an essential surface of type $C(J; n(k), \ldots, n(1))$ in $M - N(L)$.*

Proof. Given a change of basis matrix W such that $W[a] = [L]$, all other such matrices are of the form WA^N for some integer N. Proposition 4.1.4 then implies that each expression of $X(N) = (WA^N)^{-1}H(WA^N)$ as a special form corresponds to an essential surface in $M - N(L)$ with the type of an essential twisted surface. Steps 1 and 2 set up $X(N)$, and the remaining steps compute the special forms for all the $X(N)$.

Lemma 2.3.5 implies that if $X(N) = \begin{pmatrix} p(N) & r(N) \\ q(N) & s(N) \end{pmatrix}$ has a special form, then $p(N)/q(N)$ has a MCFE. This then implies that $|p(N)/q(N)| < 1$ and hence $|p(N)| < |q(N)|$. Step 3 lists all such $X(N)$. This is a finite list since $X(N) = A^{-N}(W^{-1}HW)A^N$ and hence $q(N)$ does not depend on N.

Fr each N such that $p(N)/q(N)$ has a MCFE $[a_1, a_2, \ldots, a_{k-1}]$ we may solve for a_k as in Lemma 2.3.6. Then depending on the parity of k, Lemma 2.3.6 gives us the associated special form for $X(N)$. Step 4 lists these special forms for $X(N)$.

Step 5 collects all the special forms for $X(N)$ for all N. Step 6 then follows from Proposition 4.1.4. □

4.4.1 Computing Framings

Let us continue with the notation of the algorithm above. Given a framing ζ for M we extend Algorithm 4.4.1 to describe the boundary curves of the essential surfaces of type $C(J; n(k), \ldots, n(1))$ in $M - N(L)$ in terms of the given framing.

Remark 4.4.1. This may be done as a direct computation which has been previously described in §6.2 of [6]. Lemma 4.3.1 however allows us to simplify this computation.

Algorithm 4.4.2. (Algorithm 4.4.1 Continued: Framing) *Let ζ be a framing for M. For each special form $P^J \dots B^{n(2)} A^{n(1)}$ listed in Step 5 of Algorithm 4.4.1, compute the coordinates of the boundary of the associated essential surface as follows:*

Step 7. *Take ξ to be the standard framing associated to the special form.*

Step 8. *Write $\zeta^{-1}\xi$ as a word ω in the letters ϕ and ψ.*

Step 9. *Find $j \in \mathbb{Z}$, the transition index, such that*

$$(E_\phi(\omega), E_\psi(\omega)) = (4j, 0) \mod (2, 3).$$

In the framing ζ, the boundary components of the essential surface of type $C(J; n(k), \dots, n(1))$ have coordinates $\langle 1, -j\rangle$, $\langle 2, 1-j\rangle$, $\langle 4, 1-j\rangle$, or $\langle 4, -1-j\rangle$ if $J = 0$, 2, $+1$, or -1 respectively.

Proof. In Step 8 we may write $\zeta^{-1}\xi$ in terms of ϕ and ψ by using the normal form for the structure of $\mathrm{Stab}([a,b])$ as a free product with amalgamation. In Step 9 we find the transition index via Lemma 4.3.1. Table 1 of [6] lists the boundary curves for twisted surfaces in terms of their corresponding standard framings. The transition index tells us how many times around the boundary of the fiber (the longitude) the boundary of the twisted surface wraps with respect to the given framing ζ. $\square$

4.5 Application to Berge Knots

In this section we will specialize Algorithm 4.4.1 to the case of knots in $\mathcal{K}_\eta$.

Let K be the Left Handed Trefoil, Right Handed Trefoil, or Figure Eight Knot. Then $X_K \cong T \times I/_\eta$ where η is $\alpha^{-1} \circ \beta^{-1}$, $\beta \circ \alpha$, or $\alpha \circ \beta^{-1}$ respectively (c.f. M_η and η^{-1} of §3.1).

Let L be a knot in $\mathcal{K}_\eta$. Thus L is a knot in S^3 that has a representative as an essential simple closed curve on the fiber of K.

4.5.1 Passing from X_L to $X_{K\cup L}$.

Let S_0 be an essential closed (orientable, connected) surface $\not\cong S^2$ in X_L. Assume S_0 has been chosen among surfaces in its isotopy class in X_L so that $|K \cap S_0|$ is minimized. Let $S = S_0 - N(K) \subset X_{K\cup L}$. Note that ∂S is a collection of meridional curves on $\partial N(K)$.

Lemma 4.5.1. *S is essential in $X_{K\cup L}$.*

Proof. We must show that S is incompressible, ∂-incompressible, and not ∂-parallel.

Assume S is compressible in $X_{K\cup L}$. Let D be a compressing disk for S. Since S_0 is incompressible, ∂D must bound a disk $E \subset S_0 \subset X_L$. E must intersect K since otherwise ∂D would bound the disk E in S contradicting that D is a compressing disk. Let $S_0' = (S_0 - N(E)) \cup D$. Since X_L is irreducible, S_0' is isotopic to S_0. But then $|S_0 \cap K| > |S_0' \cap K|$ contradicting the minimality assumption.

Assume S is ∂-parallel. Then S is either a torus parallel to $\partial N(L)$, a torus parallel to $\partial N(K)$, or an annulus parallel into $\partial N(K)$. If S is parallel to $\partial N(L)$ then $S = S_0$, and so S_0 is ∂-parallel in X_L. If S is parallel to $\partial N(K)$, then $S = S_0$, and so S is compressible in X_L. If S is an annulus parallel into $\partial N(K)$, then since ∂S is a collection of meridional curves, $S \cong S^2$ in X_L. These all contradict our assumptions on S.

Assume S is ∂-compressible, incompressible, and not ∂-parallel. Since the boundary components of $X_{K \cup L}$ are tori, S must be an annulus. Since ∂S is a collection of meridional curves, $S \cong S^2$ in X_L contradicting our assumption otherwise. $\square$

4.5.2 Specializing Algorithm 4.4.1

Let K, η, and L be as above. Let $H \in \mathrm{SL}_2(\mathbb{Z})$ be the matrix associated to η so that H is $A^{-1}B^{-1}$, BA, or AB^{-1} if K is the Left Handed Trefoil, Right Handed Trefoil, or Figure Eight Knot respectively. Note that η is also the corresponding standard meridional framing.

We specialize Algorithm 4.4.1 and its continuation Algorithm 4.4.2 to find closed essential surfaces in the complements of knots in $\mathcal{K}_\eta$.

Algorithm 4.5.2. *Let K and η be as above. Let $L \in \mathcal{K}_\eta$ be given by the ordered pair $(x, y) \in \mathbb{Z}^2$. The following steps give a procedure to list all closed essential surfaces in X_L.*

Step 1. *Choose a change of basis matrix $W \in \mathrm{SL}_2(\mathbb{Z})$ so that $W \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$.*

Step 2. *Let $X(N) = (WA^N)^{-1} H (WA^N)$.*

Step 3. *List all N such that $X(N) = \begin{pmatrix} p(N) & r(N) \\ q(N) & s(N) \end{pmatrix}$ where $|p(N)| < |q(N)|$.*

Step 4. *For each N listed in Step 3,*

- *obtain the MCFEs $\overline{a} = [a_1, a_2, \ldots, a_{k-1}]$ for $p(N)/q(N)$ such that k is even, and*
- *find a_k such that $[a_1, a_2, \ldots, a_{k-1}, a_k]$ is a continued fraction expansion for $r(N)/s(N)$.*

Step 5. *For each N of Step 3 and for each minimal continued fraction expansion $\overline{a} = [a_1, a_2, \ldots, a_{k-1}, a_k]$ obtained in Step 4, list those such that*

$$\sigma(\overline{a}) = E(H).$$

Step 6. *For each $\overline{a} = [a_1, a_2, \ldots, a_{k-1}, a_k]$ listed in Step 5 there is a closed essential surface of type $\widehat{C}(0; a_1, a_2, \ldots, a_{k-1}, a_k)$ in X_L.*

Proof. Assume S_0 is a closed essential surface in X_L. We may isotop S_0 to intersect K minimally. Let $S = S_0 - N(K)$. Then by Lemma 4.5.1 S is essential in $X_{K \cup L}$.

Note that S has meridional boundary on $\partial N(K)$ and is thus transverse to the induced fibration on $\partial N(K)$. From this and that η (for each of the three choices) does not fix the isotopy class of any essential curve on T, Proposition 4.2.6 implies S is isotopic to a surface of type $C(J; n(k), \ldots, n(1))$. Since S is essential, Proposition 4.1.1 implies that $|n(i)| \geq 2$ for $i = 2, \ldots, k$. Since the components of ∂S are meridional curves, by Corollary 4.3.2 $J = 0$ and $\sum_{i=1}^{k} n(i) = E(H)$.

Therefore, to every closed essential surface in X_L there is a surface of type $C(0; n(k), \ldots, n(1))$ in $X_{K \cup L}$ such that $\sum_{i=1}^{k} n(i) = E(H)$. Algorithm 4.4.1 lists all surfaces of type $C(J; n(k), \ldots, n(1))$ that are essential in $X_{K \cup L}$.

The meridional conditions $J = 0$ and $\sum_{i=1}^{k} n(i) = E(H)$ are then simple to check.

We now explain the shortcuts where the algorithm first diverges from Algorithm 4.4.1 at Step 4.

Since S must have the type of an essential twisted surface with meridional boundary components, $J = 0$ and hence in Step 4 we take the continued fraction expansions $[a_1, a_2, \ldots, a_{k-1}]$ where k is even.

At the end of Step 4, we may conclude that $X(N) = P^J B^{a_1} \ldots B^{a_{k-1}} A^{a_k}$ where $J = 0$ or $J = 2$. The framing condition of Step 5 eliminates the latter case. If $J = 2$, this special form has standard framing $\xi = \phi^2 \beta^{a_1} \ldots \beta^{a_{k-1}} \alpha^{a_k}$ and since $X(N)$ is conjugate to H which has framing η, Lemma 4.3.1 states that

$$(E_\phi(\xi\eta^{-1}), E_\psi(\xi\eta^{-1})) = (4j, 0) \mod (2, 3).$$

But since $\sum a_i = E(H)$,

$$E_\phi(\eta) = E_\psi(\eta) = E_\phi(\xi) - 2 = E_\psi(\xi).$$

Thus

$$(E_\phi(\xi\eta^{-1}), E_\psi(\xi\eta^{-1})) = (-2, 0),$$

and we get a contradiction. Therefore $J = 0$.

As in Step 5 of Algorithm 4.4.1, having a special form $B^{a_1} \ldots B^{a_{k-1}} A^{a_k}$ for $X(N)$ yields an essential surface of type $C(0; a_1, \ldots, a_k)$. The conditions of the current Step 5 imply that the boundary of this surface is meridional.

As stated in Step 6, the surface then caps off to a closed surface $\widehat{C}(0; a_1, \ldots, a_k)$ under the meridional filling of the boundary component of $X_{K\cup L}$ corresponding to K which is essential by Theorem 4.1.2.

$\square$

4.6 Complements of Essential Surfaces

Let $L \in \mathcal{K}_\eta$ be a knot on the fiber of K, the Left or Right Handed Trefoil or the Figure Eight Knot, in S^3. If S is an essential surface of type $C(0; n(k), \ldots, n(1))$ in $X_{K\cup L}$ then it must be that $|n(1)| \leq 1$. If $|n(1)| \geq 2$, then by Proposition 2.5.1 of [6] S would be isotopic to a surface of type $C(0; n(k), \ldots, n(1))$ that is essential in X_K. By the computations of [6] (see Table 2 of [6]) no such surfaces exist in the complement of K.

Let S_0 be a closed essential surface in X_L. Thus if S_0 is of the type $\widehat{C}(0; n(k), \ldots, n(1))$ then $|n(1)| \leq 1$. Note that S_0 must compress under the trivial filling of X_L.

Proposition 4.6.1. *Let K, L, and S_0 be as above.*

- *If S_0 is of type $\widehat{C}(0; n(k), \ldots, n(2), \pm 1)$, then S bounds a handlebody in S^3 and there exists an embedded annulus from S to a longitude of L.*
- *If S_0 is of type $\widehat{C}(0; n(k), \ldots, n(2), 0)$, then S does not bound a handlebody in S^3 and there exists an embedded annulus from S to the meridian of L.*

Proof. These annuli are apparent from the pictures Figures 4.4 and 4.5 of the stacking of the twisted saddle $C_{a,\pm 1}$ or $C_{a,0}$ upon its predecessor.

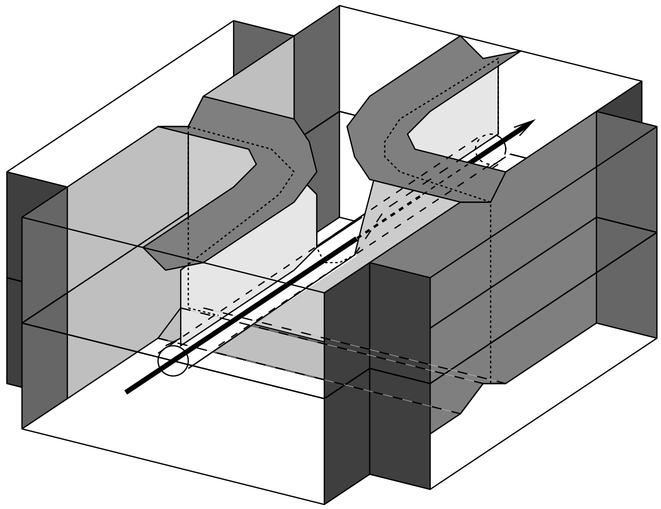


Figure 4.4: Longitudinal annulus in $C(0; n(k), \ldots, n(2), \pm 1)$.

That $\widehat{C}(0; n(k), \ldots, n(2), \pm 1)$ bounds a handlebody in S^3 and $\widehat{C}(0; n(k), \ldots, n(2), \pm 1)$ does not is harder to see.

Case 1. If the surface $\widehat{C}(0; n(k), \ldots, n(2), n(1))$ which is essential in X_L were to bound a handlebody in S^3, then L must be contained in the handlebody. Therefore we will focus on the component containing L.

Consider the essential surface $C(0; n(k), \ldots, n(2), \pm 1)$ with meridional boundary in $X_{K \cup L}$ and the corresponding surface $\widehat{C}(0; n(k), \ldots, n(2), \pm 1)$ in S^3 that is disjoint from L but intersects K four times. Filling the boundary component of $X_{K \cup L}$ corresponding to L trivially (or "forgetting" L), $C(0; n(k), \ldots, n(2), \pm 1)$ is boundary compressible in $X_K \subset S^3$. After the first two boundary compressions, the boundary of the resulting surface has two components that bound disks on $\partial X_{K \cup L}$. Thus the first two boundary compressions can be viewed as pushing $\widehat{C}(0; n(k), \ldots, n(2), \pm 1)$ through K so as to be disjoint from K. See Figures 4.6 (a)-(d) for the boundary compres-

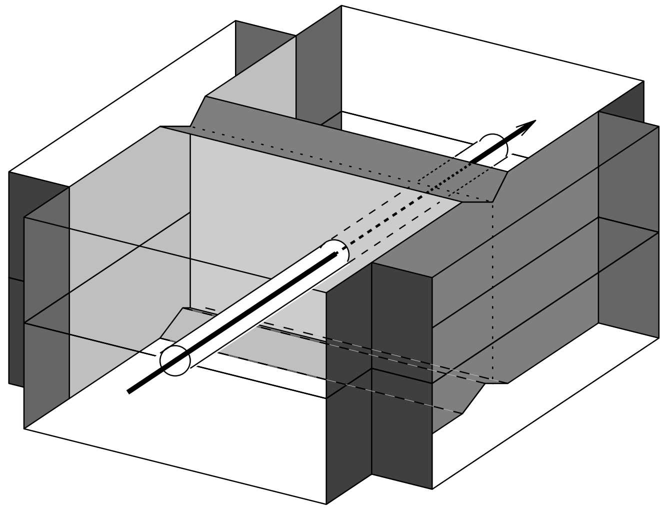


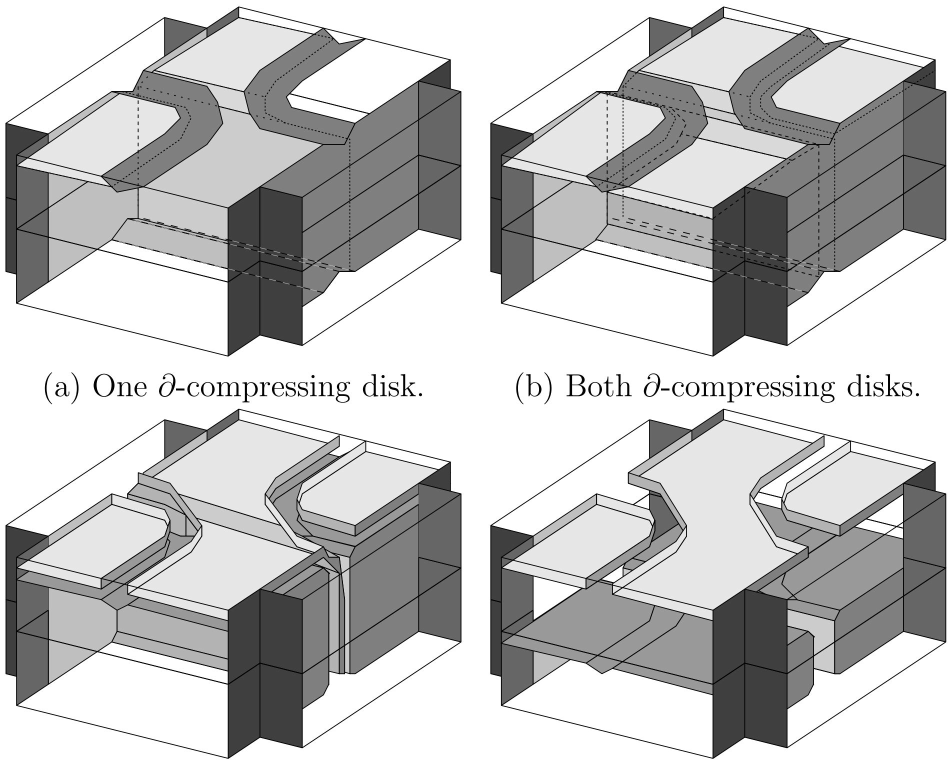
Figure 4.5: Meridional annulus in $C(0; n(k), \ldots, n(2), \pm 1)$.

sions of $C(0; n(k), \ldots, n(2), \pm 1)$. Capping off the boundary components with disks renders the surface in the first block isotopic to a disk with boundary consisting of two arcs on $\partial T \times \{1\}$ and the two arcs $\alpha^{-n(1)}(b_+)$ and $\alpha^{-n(1)}(b_-)$. (Recall $n(1) = \pm 1$.) Similarly the surface in the last block is isotopic to the disk with boundary consisting of two arcs on $\partial T \times \{1\}$ and the two arcs b_+ and b_-. See Figure4.7. Viewed in the blocks of

$$X_K = T \times I/_{\alpha^{\pm 1}} T \times I/_{\beta^{n(2)}} \ldots T \times I/_{\beta^{n(k)}},$$

this effectively adds to each of the middle $k-2$ twisted saddles two vertical disks each parallel into $\partial T \times I$; one with vertical edges $\partial a_+ \times I$ and the other with vertical edges $\partial a_- \times I$. See Figures 4.8 (a) and (b).

After this isotopy of the surface through K, we can see it bounding a handlebody as it sits in X_K. Now that the twisted saddles have sides, there is a compressing disk on the fiber joining the top of even blocks to the bottom



(a) One ∂-compressing disk. (b) Both ∂-compressing disks.

(c) After two ∂-compressions. (d) After an isotopy

Figure 4.6: Boundary compressions of $C(0; n(k), \ldots, n(2), \pm 1)$.

of odd blocks, i.e. the fiber $T \times \{1\}$ of the ith block for $2 \leq i \leq k-2$ even. This compressing disk is visible in $T \times \{1\}$ of Figure 4.8 (a) and in $T \times \{0\}$ of Figure 4.8 (b).

Compressing the surface along these $\frac{1}{2}k - 1$ disks separates the surface into $\frac{1}{2}k$ tori. There is one torus in each pair of an odd numbered block with its successive even numbered block. These tori bound solid tori. Undoing the $\frac{1}{2}k - 1$ compressions by attaching 1-handles to these solid tori and reversing the isotopies, we conclude that $\widehat{C}(0; n(k), \ldots, n(2), \pm 1)$ bounds a handlebody.

Case 2. Consider the surface $\widehat{C}(0; n(k), \ldots, n(2), 0) \subset S^3$ and its corresponding surface $C(0; n(k), \ldots, n(2), 0) \subset X_K$ in the complement of L. Let H

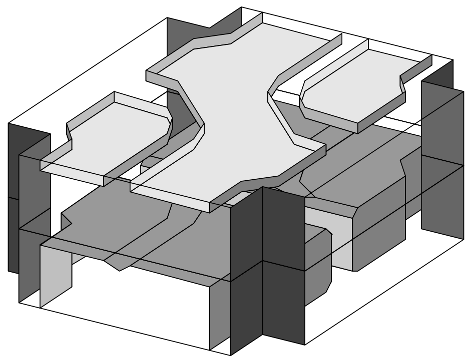
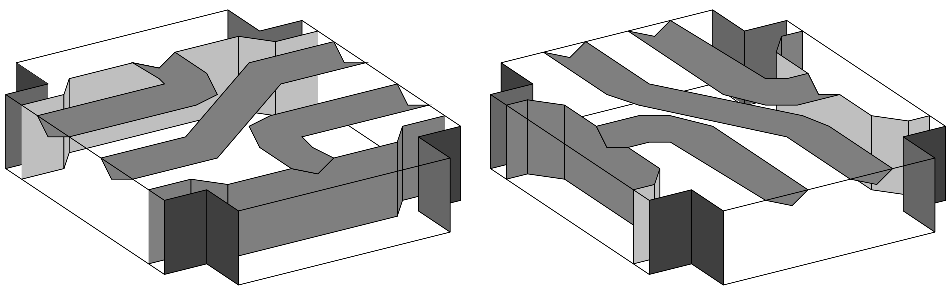


Figure 4.7: Adding sides to the first and last block.



(a) (b)

Figure 4.8: Adding sides to twisted saddles.

be the $\mathrm{SL}_2(\mathbb{Z})$ monodromy of the punctured torus bundle X_K. As in Remark 4.1.1, H is conjugate to $B^{n(k)}A^{n(k-1)}\dots A^{n(3)}B^{n(2)}A^0$. Therefore

$$\begin{aligned}
[H] &= [B^{n(k)}A^{n(k-1)}\dots A^{n(3)}B^{n(2)}A^0] \\
&= [A^{n(k-1)}\dots A^{n(3)}B^{n(2)}A^0B^{n(k)}] \\
&= [A^{n(k-1)}\dots A^{n(3)}B^{n(2)+n(k)}] \\
&= [P^{-1}A^{n(k-1)}\dots A^{n(3)}B^{n(2)+n(k)}P] \\
&= [P^{-1}PB^{n(k-1)}\dots B^{n(3)}A^{n(2)+n(k)}] \quad = [B^{n(k-1)}\dots B^{n(3)}A^{n(2)+n(k)}].
\end{aligned}$$

Since $|n(i)| \geq 2$ for $i = 3, \dots, k-1$ and the exponent sum is no different, this corresponds to another essential surface $C(0; n(k-1), \dots, n(3), n(2)+n(k))$

with meridional boundary in X_K in the complement of another level knot L'. This surface $C(0; n(k-1), \ldots, n(3), n(2)+n(k))$ is obtained by compressing along the "obvious" compressing disk for $C(0; n(k), \ldots, n(2), 0)$ forgetting L. Note that L' must be contained in the component of $X_K - N(C(0; n(k-1), \ldots, n(3), n(2)+n(k)))$ that does not come from the component of $X_K - N(C(0; n(k), \ldots, n(2), 0))$ that contained L. Also note that $n(2)+n(k) = \pm 1$ or 0.

Let us assume that $\widehat{C}(0; n(k), \ldots, n(2), 0)$ bounds a handlebody V to the side containing L in S^3. The other side must be incompressible. Note that $\widehat{C}(0; n(k), \ldots, n(2), 0) \not\cong S^2$ since there is a properly embedded separating disk (the compressing disk) that L intersects once. The compression yielding $\widehat{C}(0; n(k-1), \ldots, n(3), n(2)+n(k))$ compresses V along a nonseparating curve to yield another handlebody V' which does not contain the new level knot L'. This contradicts that $\widehat{C}(0; n(k-1), \ldots, n(3), n(2)+n(k))$ is essential in $X_{L'}$ unless $\widehat{C}(0; n(k-1), \ldots, n(3), n(2)+n(k)) \cong S^2$.

If $\widehat{C}(0; n(k-1), \ldots, n(3), n(2)+n(k)) \cong S^2$, then $k = 4$ so that the surface is $\widehat{C}(0; n(3), n(2)+n(4))$. However, one may check that there are no special forms $B^{n'(2)}A^{n'(1)}$ where $|n'(2)| \geq 2$ and $|n'(1)| \leq 1$ that are conjugate to H. (In fact one may check that there are no special forms $B^{n'(4)}A^{n'(3)}B^{n'(2)}A^{n'(1)}$ where $|n'(i)| \geq 2$ for $i = 2, 3, 4$ and $|n'(1)| \leq 1$ that are conjugate to H either.)

$\square$

4.7 Examples

Let K be the Left Handed Trefoil. Let $\eta = \alpha^{-1} \circ \beta^{-1}$ and $H = A^{-1}B^{-1}$. Here we give some examples of the application of Algorithm 4.5.2 to knots in $\mathcal{K}_\eta = \mathcal{K}_{\alpha^{-1}\circ\beta^{-1}}$, the knots on the fiber of the Left Handed Trefoil.

4.7.1 Genus 2 surfaces and the knots $\frac{1+2z}{4z} \in \mathcal{K}_\eta$.

Let $L \in \mathcal{K}_\eta$ be a knot with slope $\frac{1+2z}{4z}$ such that $z \geq 3$ is an integer. We apply Algorithm 4.5.2.

Step 1. Choose $W = \begin{pmatrix} 1+2z & -z \\ 4z & 1-2z \end{pmatrix}$.

Step 2. $X(N) = (WA^N)^{-1}H(WA^N)$.

$$X(N) = \begin{pmatrix} 3z - 6z^2 + (1+12z^2)N & 1 - 3z + 3z^2 - (1 - 6z + 12z^2)N + (1+12z^2)N^2 \\ -(1+12z^2) & 1 - 3z + 6z^2 - (1+12z^2)N \end{pmatrix}$$

Step 3. $q(N) = -(1+12z^2)$ and $p(N) = 3z - 6z^2 + (1+12z^2)N$. Therefore $|p(N)| < |q(N)|$ if $N = 0$ or $N = -1$.

Step 4. Recall that we are assuming $z \geq 3$.

($N = 0$.) $p(0)/q(0) = \frac{-3z+6z^2}{1+12z^2}$ has SCFE $[2, 1-z, 2, -1, z-1, -2]$. We have the following MCFEs and determine their corresponding a_k:

- $[2, -z, -2, -2, z, 2]$, $a_k = +1$
- $[2, -z, -2, -3, \underbrace{-2, \ldots, -2}_{z-2}, -3]$, $a_k = 0$
- $[2, 1-z, 3, z, -2]$, $a_k = 0$
- $[2, 1-z, 3, z+1, 2]$, $a_k = +1$

- $[3, \underbrace{2, \ldots, 2}_{z-2}, 3, \underbrace{-2, \ldots, -2}_{z-1}, -3]$, $a_k = 0$
- $[3, \underbrace{2, \ldots, 2}_{z-2}, 4, z, -2]$, $a_k = 0$
- $[3, \underbrace{2, \ldots, 2}_{z-2}, 4, z+1, 2]$, $a_k = +1$

$(N = -1)$. $p(-1)/q(-1) = \frac{-1-3z-6z^2}{1+12z^2}$ has SCFE $[-1, 1, 1-z, 2, -1, z-1, -2]$.

We have the following MCFEs and determine their corresponding a_k:

- $[-2, -z-1, -2, -3, \underbrace{-2, \ldots, -2}_{z-2}, -3]$, $a_k = -1$
- $[-2, -z-1, -2, -2, z, 2]$, $a_k = 0$
- $[-2, -z-1, -2, -2, z-1, -2]$, $a_k = -1$
- $[-2, -z, 2, \underbrace{-2, \ldots, -2}_{z-1}, -3]$, $a_k = -1$
- $[-2, -z, 3, z+1, 2]$, $a_k = 0$
- $[-2, -z, 3, z, -2]$, $a_k = -1$

Note that some of these MCFEs have k odd.

Step 5. Since $E(H) = E(A^{-1}B^{-1}) = -2$, we get just one list $\{a_1, \ldots, a_{k-1}, a_k\}$ such that k is even and $\sum_{i=1}^{k} a_i = -2$:

$$\{-2, -z, 3, z, -2, -1\}$$

Step 6. The knot L given by $(1+2z, 4z)$ for $z \geq 3$ contains the closed essential surface $\widetilde{C}(0; -2, -z, 3, z, -2, -1)$ in its complement. Furthermore, this is the only one.

One can actually show that a knot in $\mathcal{K}_\eta$ with slope $\frac{1+2z}{4z}$ for $|z| \geq 2$ has the surface $\widetilde{C}(0; -2, -z, 3, z, -2, -1)$ as the only closed essential surface in its complement. Moreover, one can show that if a knot in $\mathcal{K}_\eta$ has a closed essential genus 2 surface in its complement, then it is one of these knots. A similar result is also true for the knots on the fiber of the Figure Eight Knot.

4.7.2 Higher genus surfaces.

By doing the reverse of the compression in Case 2 of Proposition 4.6.1 we can construct many more knots with close essential surfaces in their complements. Using the above example as a model, we obtained the closed essential surface $\widetilde{C}(0; -2, -z, 3, z, -2, -1)$ in the complement of the knot $\frac{1+2z}{4z} \in \mathcal{K}_\eta$. Thus

$$\begin{aligned}
(A^{-N}W^{-1})A^{-1}B^{-1}(WA^N) &= B^{-2}A^{-z}B^3A^zB^{-2}A^{-1} \\
&= B^{-2}A^{-z}B^3A^zB^{-2}A^{z'-1}B^0A^{-z'} \\
A^{-z'}(A^{-N}W^{-1})A^{-1}B^{-1}(WA^N)A^{z'} &= A^{-z'}B^{-2}A^{-z}B^3A^zB^{-2}A^{z'-1}B^0 \\
P^{-1}A^{-z'}(A^{-N}W^{-1})A^{-1}B^{-1}(WA^N)A^{z'}P &= P^{-1}A^{-z'}B^{-2}A^{-z}B^3A^zB^{-2}A^{z'-1}B^0P \\
(P^{-1}A^{-z'}A^{-N}W^{-1})A^{-1}B^{-1}(WA^NA^{z'}P) &= B^{-z'}A^{-2}B^{-z}A^3B^zA^{-2}B^{z'-1}A^0.
\end{aligned}$$

Since $W = \begin{pmatrix} 1+2z & -z \\ 4z & 1-2z \end{pmatrix}$ and $N = 0$,

$$WA^NA^{z'}P = \begin{pmatrix} -z-z'-2zz' & -1-2z \\ 1-2z-4zz' & -4z \end{pmatrix}.$$

Hence if $|z'| \geq 2$ and $|z'-1| \geq 2$ (and $|z| \geq 2$), the knot $\frac{z+z'+2zz'}{-1+2z+4zz'} \in \mathcal{K}_\eta$ has in its complement the closed essential surface $\widehat{C}(0; -z', -2, -z, 3, z, -2, z'-1, 0)$ of genus 3.

This process may be repeated to construct knots in $\mathcal{K}_\eta$ whose complements contain a closed essential surface of genus g for any $g \geq 2$. There are closed essential surfaces in the complements of knots in $\mathcal{K}_\eta$, however, that are not constructed in this manner.

4.7.3 A small knot

As seen in the following chapter there are many small knots in $\mathcal{K}_\eta$. Here we give an example of the algorithm applied to a simple knot (which is not a torus knot) to show it is small.

Let $L \in \mathcal{K}_\eta$ be the knot $-\frac{3}{2} \in \mathcal{K}_\eta$. We apply Algorithm 4.5.2.

Step 1. Choose $W = \begin{pmatrix} -3 & 1 \\ 2 & -1 \end{pmatrix}$.

Step 2. $X(N) = (WA^N)^{-1}H(WA^N)$.

$$X(N) = \begin{pmatrix} -7-19N & 3+15N+19N^2 \\ -19 & 8+19N \end{pmatrix}$$

Step 3. $q(N) = -19$ and $p(N) = -7-19N$. Therefore $|p(N)| < |q(N)|$ if $N = 0$ or $N = -1$.

Step 4.

($N = 0$.) $p(0)/q(0) = \frac{7}{19}$ has SCFE $[2, -1, 2, -2]$ and two MCFEs of odd length: $[3, 4, 2]$ and $[3, 3, -2]$.

For $[3, 4, 2]$:

$$\begin{aligned} [3,4] &= (r(0) + p(0)a_4)/(s(0) + q(0)a_4) \\ \Rightarrow \frac{4}{13} &= \frac{3-7a_4}{8-19a_4} \\ &\Longrightarrow a_4 = 1 \end{aligned}$$

For $[3, 3, -2]$

$$\begin{aligned} [3,3] &= (r(0) + p(0)a_4)/(s(0) + q(0)a_4) \\ \Rightarrow \frac{3}{8} &= \frac{3-7a_4}{8-19a_4} \\ &\Longrightarrow a_4 = 0 \end{aligned}$$

$(N = -1)$. $p(-1)/q(-1) = -\frac{12}{19}$ has SCFE $[-1, 1, -1, 2, -2]$ and no MCFE of odd length.

Step 5. We have two lists:

$$\{3, 4, 2, 1\} \text{ and } \{3, 3, -2, 0\}.$$

Their sums are 10 and 4 respectively. Neither equals $E(H) = -2$.

Step 6. Since we have no lists satisfying all the criteria, there are no closed essential surfaces in the complement of L.

Chapter 5

Volumes and Surfaces

Both estimates on volumes of knots L in $\mathcal{K}$ and the computation of closed essential surfaces in the complements of such knots are related to continued fraction expansions of rational numbers. Similar relationships between continued fraction expansions and essential surfaces have appeared in [10] and [7].

In this chapter we apply Algorithm 4.4.1 to knots $L \in \mathcal{K}_\eta$ in general. Given the slope of L, we exhibit an equation in terms of the SCFE for the slope whose solutions correspond to all closed essential surfaces in the complement of L. Because estimates on $\mathrm{vol}(L)$ (if L is hyperbolic) may be obtained from continued fraction expansions for its slope, we are able to give lower bounds for the genera of closed essential surfaces in X_L in terms of lower bounds on $\mathrm{vol}(L)$. Furthermore we can construct knots in $\mathcal{K}_\eta$ with arbitrarily many distinct closed essential surfaces in their complements. Contrasting this, we can use similar techniques to construct small knots in $\mathcal{K}_\eta$ and even show that $\mathcal{K}_\eta$ contains small knots of arbitrarily large volume.

Throughout this chapter we work with $\eta = \alpha^{-1} \circ \beta^{-1}$, hence $\mathcal{K}_\eta$ are the knots that lie on the fiber of the Left Handed Trefoil. Similar results can be obtained for both the families of knots that lie on the fiber of the Right Handed Trefoil and those that lie on the Figure Eight Knot.

5.1 General Setup.

We apply Algorithm 4.5.2 to the knots $L \in \mathcal{K}_\eta$ in general. Throughout this section when $N = \pm 1$, the symbol $\pm$ agrees with the sign of N.

Let $[\overline{b}] = [b_1, b_2, \ldots, b_k]$ be the SCFE for the slope $\frac{x}{y}$ of the knot L. Recall that only b_1 may be 0 and consecutive coefficients do not have the same sign. We also assume $\frac{x}{y} \neq 0, \pm 1$, or $\frac{1}{0}$ since these slopes all correspond to the unknot.

Theorem 5.1.1. *For L as above, every closed essential surface in its complement corresponds to a solution of one of the following equations:*

1. *If $b_1 \neq 0$ or 1,*

$$0 = \sum_{i \in I} -b_i + \sum_{j \in J} b_j + \begin{cases} 0 & \textit{if } 1 \in I \\ -1 & \textit{otherwise} \end{cases}$$

   where I and J are subsets of $\{1, \ldots, k\}$ each not containing consecutive integers and $1 \notin I \cap J$.

2. *If $b_1 = 0$ and $b_2 \neq -1$,*

$$0 = \sum_{i \in I} -b_i + \sum_{j \in J} b_j + \begin{cases} 0 & \textit{if } 2 \in J \\ -1 & \textit{otherwise} \end{cases}$$

   where I and J are subsets of $\{2, \ldots, k\}$ each not containing consecutive integers and $2 \notin I \cap J$.

3. *If* $b_1 = 0$ *and* $b_2 = -1$,

$$0 = \sum_{i \in I} -b_i + \sum_{j \in J} b_j + \begin{cases} 0 & \textit{if } 3 \in J \\ -1 & \textit{otherwise} \end{cases}$$

where I *and* J *are subsets of* $\{3, \ldots, k\}$ *each not containing consecutive integers and* $3 \notin I \cap J$.

4. *If* $b_1 = 1$,

$$0 = \sum_{i \in I} -b_i + \sum_{j \in J} b_j + \begin{cases} 0 & \textit{if } 2 \in I \\ -1 & \textit{otherwise} \end{cases}$$

where I *and* J *are subsets of* $\{2, \ldots, k\}$ *each not containing consecutive integers and* $2 \notin I \cap J$.

For the resulting surface to be essential we have the extra following conditions on I *and* J.

1. *If* $b_1 \neq 0$ *or* 1, *then if* $b_l = \pm 1$ *for* $l \geq 2$, *then* $\{l-1, l, l+1\} \cap I \neq \emptyset$ *and* $\{l-1, l, l+1\} \cap J \neq \emptyset$. *If* $b_1 = -1$ *then either* $1 \in I$ *or* $\{1, 2\} \cap J \neq \emptyset$. *If* $b_1 = 2$ *then either* $1 \in J$ *or* $\{1, 2\} \cap I \neq \emptyset$.

2. *If* $b_1 = 0$ *and* $b_2 \neq -1$, *then if* $b_l = \pm 1$ *for* $l \geq 3$, *then* $\{l-1, l, l+1\} \cap I \neq \emptyset$ *and* $\{l-1, l, l+1\} \cap J \neq \emptyset$. *If* $b_2 = -2$ *then either* $2 \in I$ *or* $\{2, 3\} \cap J \neq \emptyset$. *If* $b_2 = 1$ *then either* $2 \in J$ *or* $\{2, 3\} \cap I \neq \emptyset$.

3. *If* $b_1 = 0$ *and* $b_2 = -1$, *then if* $b_l = \pm 1$ *for* $l \geq 4$, *then* $\{l-1, l, l+1\} \cap I \neq \emptyset$ *and* $\{l-1, l, l+1\} \cap J \neq \emptyset$. *If* $b_3 = 1$ *then either* $3 \in J$ *or* $\{3, 4\} \cap I \neq \emptyset$.

4. *If* $b_1 = 1$, *then if* $b_l = \pm 1$ *for* $l \geq 3$, *then* $\{l-1, l, l+1\} \cap I \neq \emptyset$ *and* $\{l-1, l, l+1\} \cap J \neq \emptyset$. *If* $b_2 = -1$ *then either* $2 \in I$ *or* $\{2, 3\} \cap J \neq \emptyset$.

Remark 5.1.1. Cases $2-4$ and the extra conditions on the sets I and J are not needed if $[\overline{b}]$ is a MCFE as well as a SCFE with $b_1 \neq 2$.

Proof. Set $W = \begin{pmatrix} x & t \\ y & u \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ where

$$\frac{x}{y} = [b_1, b_2, \ldots, b_k] \text{ and}$$
$$\frac{t}{u} = [b_1, b_2, \ldots, b_{k-1}].$$

Then by Lemma 2.3.5 W may be written as

$$\pm B^{b_1} A^{b_2} \ldots B^{b_k} \text{ or } \pm BABA^{b_1} B^{b_2} \ldots B^{b_k}$$

depending on the parity of k.

Let

$$X(N) = (WA^N)^{-1} A^{-1} B^{-1} (WA^N) = \begin{pmatrix} p(N) & q(N) \\ r(N) & s(N) \end{pmatrix} A^N$$

for $N \in \mathbb{Z}$. Then $X(N)$ may be written as

$$(\pm A^{-N} B^{-b_k} \ldots A^{-b_2} B^{-b_1}) A^{-1} B^{-1} (\pm B^{b_1} A^{b_2} \ldots B^{b_k} A^N)$$
$$= A^{-N} B^{-b_k} \ldots A^{-b_2} B^{-b_1} A^{-1} B^{b_1 - 1} A^{b_2} \ldots B^{b_k} A^N$$

or

$$(\pm A^{-N} B^{-b_k} \ldots B^{-b_2} A^{-b_1} B^{-1} A^{-1} B^{-1}) A^{-1} B^{-1} (\pm B^1 A^1 B^1 A^{b_1} B^{b_2} \ldots B^{b_k} A^N)$$
$$= A^{-N} B^{-b_k} \ldots B^{-b_2} A^{-b_1} B^{-1} A^{b_1 - 1} B^{b_2} \ldots B^{b_k} A^N$$

depending on the parity of k. In either case, note that

$$p(N)/q(N) = [0, -N, -b_k, \ldots, -b_2, -b_1, -1, b_1 - 1, b_2, \ldots, b_k].$$

Define $[\overline{x(N)}] = [0, -N, -b_k, \ldots, -b_2, -b_1, -1, b_1 - 1, b_2, \ldots, b_k]$.

According to Algorithm 4.5.2 for every MCFE $[\overline{x_m}] = [a_1, a_2, \ldots, a_l]$ of $p(N)/q(N)$ of odd length such that $N' = -2 - \sigma(\overline{x_m})$ and $[a_1, a_2, \ldots, a_l, N']$ is a continued fraction expansion for $r(N)/s(N)$ for some N there is a closed essential surface in the complement of L. Since every MCFE of a rational number is derived from its SCFE (as noted in §2.3.1), we need to obtain the SCFE $[\overline{x_s(N)}]$ for $p(N)/q(N)$. We determine the SCFE in §5.1.1 and the MCFEs corresponding to essential surfaces in §5.1.2.

5.1.1 SCFE for $p(N)/q(N)$.

First recall that if $p(N)/q(N)$ is to have a MCFE then its absolute value must be less than 1. Since $p(N)/q(N) = [\overline{x(N)}]$,

$$p(N)/q(N) = \cfrac{1}{0 - \cfrac{1}{-N - [\overline{b'}]}} = N + [\overline{b'}]$$

where $[\overline{b'}] = [-b_k, \ldots, -b_2, -b_1, -1, b_1 - 1, b_2, \ldots, b_k]$.

Lemma 5.1.2. *$|[\overline{b'}]| < 1$ and $\mathrm{sign}([\overline{b'}]) = \mathrm{sign}(-b_k)$.*

Proof. We exhibit the SCFE for $[\overline{b'}]$. The first coefficient will be nonzero and hence imply the conclusion of the lemma.

By move (CF 1),

$$[\overline{b'}] \mapsto [\overline{b'_1}] = [-b_k, \ldots, -b_2, -b_1 + 1, b_1, b_2, \ldots, b_k].$$

This is a SCFE unless $b_1 = 0$ or 1.

Case $b_1 = 0$. By move (CF 1)

$$\begin{aligned}[\overline{b'_1}] &= [-b_k, \ldots, -b_2, 1, 0, b_2, \ldots, b_k] \\ &\mapsto [-b_k, \ldots, -b_3, -b_2, b_2 + 1, b_3, \ldots, b_k] = [\overline{b'_2}]\end{aligned}$$

which is a SCFE unless $b_2 = -1$ or 0. Since $[\overline{b}]$ is a SCFE, b_2 cannot be 0. Furthermore $k > 1$ since $[\overline{b}] \neq 0$.

If $b_2 = -1$ then by move (CF 1)

$$\begin{aligned}[\overline{b'_2}] &= [-b_k, \ldots, -b_3, 1, 0, b_3, \ldots, b_k] \\ &\mapsto [-b_k, \ldots, -b_4, -b_3, b_3 + 1, b_4, \ldots, b_k] = [\overline{b'_3}]\end{aligned}$$

which is a SCFE unless $b_3 = -1$ or 0. Since $[\overline{b}]$ is a SCFE and $b_2 = -1$ neither of these two situations can occur. Furthermore if $k = 2$ then $[\overline{b}] = [0, -1] = 1$ contrary to our assumption on $\frac{x}{y}$.

Case $b_1 = 1$. By move (CF 1)

$$\begin{aligned}[\overline{b'_1}] &= [-b_k, \ldots, -b_2, 0, 1, b_2, \ldots, b_k] \\ &\mapsto [-b_k, \ldots, -b_3, -b_2 + 1, b_2, b_3, \ldots, b_k] = [\overline{b'_4}]\end{aligned}$$

which is a SCFE unless $b_2 = 0$ or 1. Since $[\overline{b}]$ is a SCFE and $b_1 = 1$ neither of these two situations can occur. Furthermore if $k = 1$ then $[\overline{b}] = 1$ contrary to our assumption on $\frac{x}{y}$.

Each of $[\overline{b'_i}]$ for $i = 1, 2, 3, 4$ has nonzero leading coefficient. One of these is the SCFE for $[\overline{b'}]$. Therefore by Lemma 2.3.1, $|[\overline{b'}]| < 1$ and $\text{sign}([\overline{b'}]) = \text{sign}(-b_k)$. □

Corollary 5.1.3. *If* $p(N)/q(N)$ *has a MCFE then either* $N = 0$ *or* $N = \pm 1$ *such that* $\text{sign}(N) = \text{sign}(b_k)$.

Proof. A rational number has a MCFE only if its absolute value is less than 1. Since $p(N)/q(N) = [0, N, \overline{b'}] = N + [\overline{b'}]$ and by Lemma 5.1.2 $|[\overline{b'}]| < 1$,

$|p(N)/q(N)| < 1$ if and only if either $N = 0$ or $N = \pm 1$ such that $\operatorname{sign}(N) \neq \operatorname{sign}([\overline{b'}])$. However, $\operatorname{sign}([\overline{b'}]) = \operatorname{sign}(-b_k) = -\operatorname{sign}(b_k)$. $\square$

Remark 5.1.2. Corollary 5.1.3 shows that we really only need to consider two values of N in Algorithm 4.5.2.

We now find the SCFE $[\overline{x_s(N)}]$ for $p(N)/q(N)$.

If $N = 0$, then for one of $i = 1, 2, 3$, or 4.

$$p(0)/q(0) = [\overline{x(0)}] = [0, 0, \overline{b'_i}] = [\overline{b'_i}]$$

is the SCFE.

If $N = \pm 1$, then for $i = 1, 2, 3$, or 4 such that $[\overline{b'_i}]$ is a SCFE

$$\begin{aligned} p(\pm 1)/q(\pm 1) = [\overline{x(\pm 1)}] = [0, \mp 1, \overline{b'_i}] &= [0, \mp 1, -b_k, \ldots, b_k] \\ &\mapsto [\pm 1, -b_k \pm 1, \ldots, b_k] = [\overline{b''_i}] \end{aligned}$$

by move (CF 1). This is a SCFE unless $b_k = 0$ or ± 1. Since $[\overline{b'_i}]$ is a SCFE, $|b_k| \geq 2$. Therefore $[\overline{b''_i}]$ is the SCFE.

Notice that the move (CF 1') is never used in obtaining $[\overline{x_s(N)}]$ from $[\overline{x(N)}]$. Hence their last two partial sums are equal. Furthermore, as a consequence of Lemma 2.3.6 there exists an N_s such that $[\overline{x(N)}, N] = [\overline{x_s(N)}, N_s]$. Therefore by Lemma 2.3.2, $N_s = N'$.

There are two extra "degenerate" cases to consider for the SCFE of $p(N)/q(N)$.

If $n = 2$, $i = 4$ (so $b_1 = 1$), and $N = 1$, then

$$[\overline{b'_4}] = [1, -b_2 + 1 + 1, b_2] = [1, -b_2 + 2, b_2]$$

is a SCFE unless $b_2 = 0, 1$, or 2. These cannot occur since $b_1 = 1$ and $[\overline{b}]$ is a SCFE.

If $n = 1$, $i = 1$, and $N = 1$, then

$$[\overline{b_1'}] = [1, -b_1 + 1 + 1, b_1] = [1, -b_1 + 2, b_1]$$

is a SCFE unless $b_1 = 0, 1$, or 2. By our assumption on $\frac{x}{y}$, $b_1 \neq 0$ or 1. However, b_1 may equal 2 in which case $\frac{x}{y} = \frac{1}{2}$. This slope corresponds to the unknot.

Therefore we obtain:

Lemma 5.1.4. *For $N = 0$ or ± 1, $p(N)/q(N)$ has SCFE $[\overline{x_s(N)}]$ where $\overline{x_s(0)} = \overline{b_i'}$ and $\overline{x_s(\pm 1)} = \overline{b_i''}$ for some $i \in \{1, 2, 3, 4\}$. Furthermore $r(N)/s(N)$ has the continued fraction expansion $[\overline{x_s(N)}, N]$.*

5.1.2 MCFEs and Surfaces.

Given the SCFE $[\overline{x_s(N)}]$ for $p(N)/q(N)$, we determine every MCFE $[\overline{x_m}]$ of odd length and N' such that both $r(N)/s(N) = [\overline{x_m}, N']$ and $\sigma(\overline{x_m}) + N' = \sigma(\overline{x(N)}) + N$.

For such an $[\overline{x_m}]$ and N', since $[\overline{x_s(N)}, N] = r(N)/s(N)$ by Lemma 5.1.4, Lemma 2.3.4 implies $\sigma(\overline{x_m}) - \sigma(\overline{x(N)}) = N - N' = \begin{cases} 0 \\ \text{sign}(b_k) \end{cases}$.

We have four cases for the four different possibilities of the SCFE $[\overline{x_s(N)}]$ depending on $[\overline{b}]$ each of which has the two cases of $N = 0$ and $N = \pm 1$. See Lemma 5.1.4.

1. If $b_1 \neq 0, 1$ then $[\overline{x_s(0)}] = [\overline{b_1'}]$ and $[\overline{x_s(\pm 1)}] = [\overline{b_1''}]$.

2. If $b_1 = 0$ and $b_2 \neq -1$ then $[\overline{x_s(0)}] = [\overline{b_2'}]$ and $[\overline{x_s(\pm 1)}] = [\overline{b_2''}]$.

3. If $b_1 = 0$ and $b_2 = -1$ then $[\overline{x_s(0)}] = [\overline{b_3'}]$ and $[\overline{x_s(\pm 1)}] = [\overline{b_3''}]$.

4. If $b_1 = 1$ then $[\overline{x_s(0)}] = [\overline{b_4'}]$ and $[\overline{x_s(\pm 1)}] = [\overline{b_4''}]$.

Case 1. $b_1 \neq 0, 1$

The SCFEs for $p(N)/q(N)$ for $N = 0, \pm 1$ are

$$[\overline{x_s(0)}] = [\overline{b_1'}] = [-b_k, \ldots, -b_2, -b_1 + 1, b_1, b_2, \ldots, b_k]$$

and

$$[\overline{x_s(\pm 1)}] = [\overline{b_1''}] = [\pm 1, -b_k \pm 1, \ldots, -b_2, -b_1 + 1, b_1, b_2, \ldots, b_k].$$

Subcase $N = 0$.

Assume $[\overline{x_m}]$ is a MCFE for $p(0)/q(0)$. Then it is obtained from $[\overline{x_s(0)}] = [\overline{b_1'}]$ by applying moves (M) and (M') to nonadjacent coefficients of $[\overline{b_1'}]$. (C.f. §2.3.1.) In particular, if a coefficient of $[\overline{b_1'}]$, then (M) or (M') must be applied to it or an adjacent coefficient to make the all the resulting coefficients of $[\overline{x_m}]$ not ± 1.

The difference of coefficient sums between $[\overline{x_m}]$ and $[\overline{b_1'}]$ is

$$\begin{aligned}
\sigma(\overline{x_m}) - \sigma(\overline{b_1'}) &= \sum_{i \in I} -3(-b_i) + \begin{cases} -3(1) & \text{if } 1 \in I \\ 0 & \text{otherwise} \end{cases} + \sum_{j \in J} -3(b_j) \\
&\qquad + \begin{cases} \operatorname{sign}(b_k) & \text{if } k \in J \\ 0 & \text{otherwise} \end{cases} \\
&= -3\left(\sum_{i \in I} -b_i + \begin{cases} 1 & \text{if } 1 \in I \\ 0 & \text{otherwise} \end{cases} + \sum_{j \in J} b_j \right) \\
&\qquad + \begin{cases} \operatorname{sign}(b_k) & \text{if } k \in J \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

where $I \subset \{1, \ldots, k-1\}$ and $J \subset \{1, \ldots, k\}$ each not containing consecutive integers and $1 \notin I \cap J$. If $k \in J$, then we take $\operatorname{sign}(b_k)$ since we need $+1$ if $b_k > 0$ and -1 if $b_k < 0$. The indexing sets I and J correspond to the coefficients to which the moves (M) and (M') are applied.

Since $\sigma(\overline{x_m}) - \sigma(\overline{x(0)}) = \begin{cases} 0 \\ \operatorname{sign}(b_k) \end{cases}$ and

$$\sigma(\overline{x_s(0)}) - \sigma(\overline{x(0)}) = \sigma(\overline{b_1'}) - \sigma(\overline{x(0)}) = 1 - -2 = +3,$$

then

$$\left.\begin{array}{c} 0 \\ \operatorname{sign}(b_k) \end{array}\right\} = \sigma(\overline{x_m}) - \sigma(\overline{x(0)})$$

$$\begin{aligned}
&= \sigma(\overline{x_m}) - \sigma(\overline{b_1'}) + \sigma(\overline{b_1'}) - \sigma(\overline{x(N)}) \\
&= -3\left(\sum_{i \in I} -b_i + \begin{cases} 1 & \text{if } 1 \in I \\ 0 & \text{otherwise} \end{cases} + \sum_{j \in J} b_j\right) \\
&\qquad + \begin{cases} \operatorname{sign}(b_k) & \text{if } k \in J \\ 0 & \text{otherwise} \end{cases} + 3 \\
&= -3\left(\sum_{i \in I} -b_i + \begin{cases} 1 & \text{if } 1 \in I \\ 0 & \text{otherwise} \end{cases} + \sum_{j \in J} b_j - 1\right) \\
&\qquad + \begin{cases} \operatorname{sign}(b_k) & \text{if } k \in J \\ 0 & \text{otherwise} \end{cases} \\
&= -3\left(\sum_{i \in I} -b_i + \sum_{j \in J} b_j + \begin{cases} 0 & \text{if } 1 \in I \\ -1 & \text{otherwise} \end{cases}\right) \\
&\qquad + \begin{cases} \operatorname{sign}(b_k) & \text{if } k \in J \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

If $k \in J$ then

$$\left.\begin{array}{c} 0 \\ \operatorname{sign}(b_k) \end{array}\right\} = -3\left(\sum_{i \in I} -b_i + \sum_{j \in J} b_j + \begin{cases} 0 & \text{if } 1 \in I \\ -1 & \text{otherwise} \end{cases}\right) + \operatorname{sign}(b_k)$$

which implies we must take $\sigma(\overline{x_m}) - \sigma(\overline{x(0)}) = \operatorname{sign}(b_k)$. Thus if $k \in J$, then $N' = N - \operatorname{sign}(b_k) = -\operatorname{sign}(b_k)$.

If $k \notin J$, then similarly we must take which implies we must take $\sigma(\overline{x_m}) - \sigma(\overline{x(0)}) = 0$. Thus if $k \notin J$, then $N' = N = 0$.

In either situation,

$$0 = -3\left(\sum_{i\in I} -b_i + \sum_{j\in J} b_j + \begin{cases} 0 & \text{if } 1 \in I \\ -1 & \text{otherwise} \end{cases}\right),$$

and hence

$$0 = \sum_{i\in I} -b_i + \sum_{j\in J} b_j + \begin{cases} 0 & \text{if } 1 \in I \\ -1 & \text{otherwise.} \end{cases}$$

Subcase $N = \pm 1$.

Assume $\overline{x_m}$ is a MCFE for $p(\pm 1)/q(\pm 1)$. Then it is obtained from $[\overline{x_s(\pm 1)}] = [\overline{b_1''}]$ by applying moves (M) and (M') to nonadjacent coefficients of $[\overline{x_s(0)}]$. (C.f. §2.3.1.) Again, if a coefficient of $[\overline{b_1''}]$ is ± 1, then move (M) or (M') must be applied to it or an adjacent coefficient.

Then the difference of coefficient sums is

$$\begin{aligned}\sigma(\overline{x_m}) - \sigma(\overline{b_1''}) &= -3(\pm 1) + \sum_{i\in I} -3(-b_i) + \begin{cases} -3(1) & \text{if } 1 \in I \\ 0 & \text{otherwise} \end{cases} + \sum_{j\in J} -3(b_j) \\ &\qquad + \begin{cases} \operatorname{sign}(b_k) & \text{if } k \in J \\ 0 & \text{otherwise} \end{cases} \\ &= -3\left(\pm 1 + \sum_{i\in I} -b_i + \begin{cases} 1 & \text{if } 1 \in I \\ 0 & \text{otherwise} \end{cases} + \sum_{j\in J} b_j\right) \\ &\qquad + \begin{cases} \operatorname{sign}(b_k) & \text{if } k \in J \\ 0 & \text{otherwise} \end{cases}\end{aligned}$$

where $I \subset \{1, \ldots, k-2, k\}$ and $J \subset \{1, \ldots, k\}$ each not containing consecutive integers, $k \in I$, and $1 \notin I \cap J$. Note that we must apply move (M) to the second coefficient $-b_k \pm 1$ in $[\overline{b_1''}]$ to account for the leading coefficient of ± 1, hence $k \in I$. Also if $k \in J$, then we take $\operatorname{sign}(b_k)$ since we need $+1$ if $b_k > 0$ and -1 if $b_k < 0$. As noted in the previous case, the indexing sets I and J correspond to the coefficients to which the moves (M) and (M') are applied.

Since $\sigma(\overline{x_m}) - \sigma(\overline{x(\pm 1)}) = \begin{cases} 0 \\ \operatorname{sign}(b_k) \end{cases}$ and

$$\begin{aligned} \sigma(\overline{x_s(\pm 1)}) - \sigma(\overline{x(\pm 1)}) &= \sigma(\overline{b_1''}) - \sigma(\overline{x(\pm 1)}) \\ &= (\pm 2 + 1) - (\mp 1 + -2) = \pm 3 + 3 \end{aligned}$$

then

$$\begin{aligned} \left.\begin{matrix} 0 \\ \operatorname{sign}(b_k) \end{matrix}\right\} &= \sigma(\overline{x_m}) - \sigma(\overline{x(\pm 1)}) \\ &= \sigma(\overline{x_m}) - \sigma(\overline{b_1''} + \sigma(\overline{b_1''}) - \sigma(\overline{x(\pm 1)}) \\ &= -3 \left(\pm 1 + \sum_{i \in I} -b_i + \begin{cases} 1 & \text{if } 1 \in I \\ 0 & \text{otherwise} \end{cases} + \sum_{j \in J} b_j \right) \\ &\qquad + \begin{cases} -\operatorname{sign}(b_k) & \text{if } k \in J \\ 0 & \text{otherwise} \end{cases} \pm 3 + 3 \\ &= -3 \left(\pm 1 + \sum_{i \in I} -b_i + \begin{cases} 1 & \text{if } 1 \in I \\ 0 & \text{otherwise} \end{cases} + \sum_{j \in J} b_j \mp 1 - 1 \right) \\ &\qquad + \begin{cases} \operatorname{sign}(b_k) & \text{if } k \in J \\ 0 & \text{otherwise} \end{cases} \\ &= -3 \left(\sum_{i \in I} -b_i + \sum_{j \in J} b_j + \begin{cases} 0 & \text{if } 1 \in I \\ -1 & \text{otherwise} \end{cases} \right) \\ &\qquad + \begin{cases} \operatorname{sign}(b_k) & \text{if } k \in J \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

If $k \in J$ then

$$\left.\begin{matrix} 0 \\ \operatorname{sign}(b_k) \end{matrix}\right\} = -3 \left(\sum_{i \in I} -b_i + \sum_{j \in J} b_j + \begin{cases} 0 & \text{if } 1 \in I \\ -1 & \text{otherwise} \end{cases} \right) + \operatorname{sign}(b_k)$$

which implies we must take $\sigma(\overline{x_m}) - \sigma(\overline{x(0)}) = \text{sign}(b_k)$. Thus if $k \in J$, then $N' = N - \text{sign}(b_k) = \pm 1 - \text{sign}(b_k) = 0$ since $\text{sign}(b_k) = N = \pm 1$.

If $k \notin J$, then similarly we must take which implies we must take $\sigma(\overline{x_m}) - \sigma(\overline{x(0)}) = 0$. Thus $N' = N = \pm 1$.

In either situation,

$$0 = -3\left(\sum_{i\in I} -b_i + \sum_{j\in J} b_j + \begin{cases} 0 & \text{if } 1 \in I \\ -1 & \text{otherwise} \end{cases}\right)$$

and hence

$$0 = \sum_{i\in I} -b_i + \sum_{j\in J} b_j + \begin{cases} 0 & \text{if } 1 \in I \\ -1 & \text{otherwise.} \end{cases}$$

The only difference between the outcomes of these two subcases is in the indexing set I. If $N = 0$ then $I \subset \{1, \ldots, k-1\}$ containing no consecutive integers. If $N = \pm 1$ then $I \subset \{1, \ldots, k-2, k\}$ containing k and no consecutive integers. (Notice that $N = 0$ if $k \notin I$ and $N = \pm 1 = \text{sign}(b_k)$ if $k \in I$.) These two conditions can be consolidated by taking $I \subset \{1, \ldots, k\}$ containing no consecutive integers. Therefore a solution to the equation

$$0 = \sum_{i\in I} -b_i + \sum_{j\in J} b_j + \begin{cases} 0 & \text{if } 1 \in I \\ -1 & \text{otherwise} \end{cases} \tag{*}$$

where I and J are subsets of $\{1, \ldots, k\}$ each not containing consecutive integers and $1 \notin I \cap J$ corresponds to an $N = 0$ or ± 1, a continued fraction $[\overline{x_m}] = p(N)/q(N)$, and an N' such that both $[\overline{x_m}, N'] = r(N)/s(N)$ and $\sigma(\overline{x_m}) + N' = \sigma(\overline{x(N)}) + N$. The continued fraction $[\overline{x_m}]$ will be minimal only if moves (M) or (M') are applied to coefficients ± 1 of $[\overline{x_s(N)}]$ or their neighbors.

A priori, such an $[\overline{x_m}]$ might not be of odd length. Following the method for calculating coefficient sums, the length of $[\overline{x_m}]$ is given by

$$\text{length}(\overline{x_m}) - \text{length}(\overline{x_s(N)}) = \sum_{i \in I}(|-b_i| - 2) + \sum_{j \in J}(|b_j| - 2)$$
$$+ \begin{cases} 0 & \text{if } 1 \in I \\ -1 & \text{otherwise} \end{cases} + \begin{cases} N & \text{if } k \in I \\ 0 & \text{otherwise.} \end{cases}$$

Because $[\overline{x_s(N)}]$ is $[\overline{b_1'}]$ when $k \notin I$ and $[\overline{b_1''}]$ when $k \in I$,

$$\text{length}(\overline{x_s(N)}) = \begin{cases} 2k & \text{if } k \notin I \\ 2k+1 & \text{if } k \in I. \end{cases}$$

Viewing this mod 2, $[\overline{x_m}]$ has odd length if and only if

$$1 - \begin{cases} 0 & \text{if } k \notin I \\ 1 & \text{if } k \in I. \end{cases} = \sum_{i \in I} -b_i + \sum_{j \in J} b_j$$
$$+ \begin{cases} 0 & \text{if } 1 \in I \\ -1 & \text{otherwise} \end{cases} + \begin{cases} 1 & \text{if } k \in I \\ 0 & \text{otherwise} \end{cases} \mod 2.$$

Therefore

$$0 = \sum_{i \in I} -b_i + \sum_{j \in J} b_j + \begin{cases} 0 & \text{if } 1 \in I \\ -1 & \text{otherwise} \end{cases} \mod 2.$$

Hence the odd length of $[\overline{x_m}]$ is implied by the solution to equation (*). The conclusion of the theorem in this case follows.

Cases 2 - 4. These last three cases follow much the same via the appropriate substitutions of $[\ldots, -b_2, -b_1 + 1, b_1, b_2, \ldots]$ by $[\ldots, -b_3, -b_2, b_2 + 1, b_3, \ldots]$, $[\ldots, -b_4, -b_3, b_3 + 1, b_4, \ldots]$, and $[\ldots, -b_3, -b_2 + 1, b_2, b_3, \ldots]$.

The conclusion of Theorem 5.1.1 then follows.

$\square$

5.2 Corollaries.

5.2.1 Genera of Surfaces.

Corollary 5.2.1. *If $L \in \mathcal{K}_\eta$ is a hyperbolic knot with* $\mathrm{vol}(L) > \mathrm{vol}(L(2n+1), \eta) > (2n+1) \cdot v$*, then X_L contains no closed essential surfaces of genus less than $\frac{1}{2}(n-1)$.*

Proof. Assume $L \in \mathcal{K}_\eta$ has volume greater than $\mathrm{vol}(L(2n+1), \eta)$. Then since volume decreases under surgery, L cannot be written as surgery on the link $L(2n+1, \eta)$. Thus if L has slope $\frac{x}{y}$, then any continued fraction expansion for $\frac{x}{y}$ must have length greater than n.

Let $[\overline{b}] = [b_1, b_2, \ldots, b_k]$ be the SCFE for $\frac{x}{y}$. Note that $k > n$. Construct $[\overline{x_s(N)}]$ as in Lemma 5.1.4.

Since $[\overline{b}]$ has length k, the length of $[\overline{x_s(N)}]$ must be at least $2k-2$, the length of $[\overline{b'_4}]$.

Since the genus of a closed essential surface is related to the length of the associated MCFE of $p(N)/q(N)$, we need to see how much shorter a MCFE can be that the SCFE from which it is obtained.

Recall that any MCFE may be obtained from the corresponding SCFE by the moves (M 1) and (M 1') on non-adjacent coefficients of the SCFE. Since length only decreases under these moves applied to a coefficient of ± 1, each of which decreases the length by 1, the length of a MCFE must be at least half the length of its corresponding SCFE. Generically, however, the length will increase.

Therefore the length of a MCFE for $p(N)/q(N)$ must be at least $\frac{1}{2}(2k-2)$ and hence greater than $k-1$. Since $k > n$, the length of a MCFE for $p(N)/q(N)$ is greater than $n-1$. If a MCFE $[\overline{x_m(N)}]$ for $p(N)/q(N)$ of length k' corresponds to a twisted surface with meridional boundary, then the genus of the closed essential surface is $\frac{1}{2}(k'+1)-1$. Since $k' \geq k$, the genus of a closed essential surface in the complement of L must be greater than $\frac{1}{2}(n-1)-1$.

$\square$

Corollary 5.2.2. *There exist knots in $\mathcal{K}_\eta$ with arbitrarily many distinct closed essential surfaces in their complements.*

Proof. Let $[\bar{b}] = [b_1, \ldots, b_k]$ be the SCFE of the slope of a knot $K_0 \in \mathcal{K}_\eta$ with a closed essential surface in its complement such that $b_1 \neq 0$ or 1. By Theorem 5.1.1 there are subsets I_0 and J_0 of $\{1, \ldots, k\}$ such that

$$0 = \sum_{i \in I_0} -b_i + \sum_{j \in J_0} + \begin{cases} 0 & \text{if } 1 \in I \\ -1 & \text{otherwise.} \end{cases}$$

For each integer $r \geq 2$, let $[\overline{a_r}] = [a_{k+1}, a_{k+2}, \ldots, a_{k+r}]$ be a SCFE and a MCFE such that $\operatorname{sign}(a_{k+1}) = -\operatorname{sign}(b_k)$. Therefore $[\bar{b}, \overline{a_r}]$ is a SCFE. Let K_r be the knot in $\mathcal{K}_\eta$ with this slope.

Let $\mathcal{R}_r$ be the collection of all subsets of $\{k+2, k+3, \ldots, k+r\}$ that do not contain consecutive integers. Then for every $R \in \mathcal{R}_r$,

$$0 = \sum_{i \in I_0 \cup R} -b_i + \sum_{j \in J_0 \cup R} + \begin{cases} 0 & \text{if } 1 \in I \\ -1 & \text{otherwise.} \end{cases}$$

Hence for every $R \in \mathcal{R}_r$ there is a closed essential surface in the complement of K_r.

Given $R, R' \in \mathcal{R}_r$ such that $R \subset R'$, then the surfaces obtained by this construction corresponding to R and R' have genera g and g' respectively such that $g < g'$. Therefore the surfaces are distinct.

As r increases, so does the cardinality of $\mathcal{R}_r$ and size of maximal sets $R \in \mathcal{R}_r$. Therefore as r increases the number of distinct closed essential surfaces in the complement of K_r increases. $\square$

5.2.2 Small Knots.

Corollary 5.2.3. *Let $\Phi_0 \geq 5$ be an integer. Let $[\overline{b}] = [b_1, b_2, \ldots, b_k]$ be a SCFE such that $b_i \equiv 0 \mod \Phi_0$ for $i = 2, \ldots, k$ and $b_1 \not\equiv 0$ or $1 \mod \Phi_0$. Then the knot $L \in K_\eta$ with slope $\frac{x}{y} = [\overline{b}]$ is small.*

Proof. Because $b_1 \neq 0$ or 1, we may apply case (1) of Theorem 5.1.1. An essential surface in the complement of L will correspond to a solution to

$$0 = \sum_{i \in I} -b_i + \sum_{j \in J} b_j + \begin{cases} 0 & \text{if } 1 \in I \\ -1 & \text{otherwise} \end{cases}$$

subject to various constraints on the sets I and J. A solution to this equation will hold true $\mod \Phi$. By choosing $b_i \equiv 0 \mod \Phi$ for $i \geq 2$, this equation becomes

$$0 \equiv \begin{cases} -b_1 & \text{if } 1 \in I \\ b_1 - 1 & \text{if } 1 \in J \\ -1 & \text{otherwise} \end{cases} \mod \Phi.$$

This equation can only hold true if $b_1 \equiv 0$ or $1 \mod \Phi$. Having chosen b_1 otherwise, this equation and hence the original equations have no solution. Therefore there can be no closed essential surfaces in the complement of L.

$\square$

Corollary 5.2.4. *The set of volumes of small knots on the fiber of the Left Handed Trefoil is unbounded.*

In other words, there are small knots of arbitrarily large volume on the fiber of the Left Handed Trefoil. (Again, a similar statement can be made for knots on the fiber of the Right Handed Trefoil and the Figure Eight Knot.)

Proof. For each integer $n \geq 2$ choose an integer $\Phi_n >> 5$ that is "sufficiently large" as at the end of the proof of Theorem 3.2.1. Consider the sequence of knots $K_n \in \mathcal{K}_\eta$ represented by the simple continued fractions of length n

$$[\overline{k_n}] = [\Phi_n + 2, -\Phi_n, +\Phi_n, -\Phi_n, +\Phi_n, \ldots, \pm\Phi_n].$$

By Theorem 3.2.1, $\mathrm{vol}(K_n) \to \infty$ as $n \to \infty$. Thus the set of volumes $\{\mathrm{vol}(K_n)\}$ is unbounded.

All the coefficients of $[\overline{k_n}]$ except the first are congruent to $0 \mod \Phi_n$. The first coefficient is $\Phi_n + 2 \equiv 2 \mod \Phi_n$ and is not congruent to 0 or 1. Since the coefficients of $[\overline{k_n}]$ alternate sign, it is a simple continued fraction. By Lemma 5.2.3, the knots K_n are all small.

□

Remark 5.2.1. The class of two-bridge knots also enjoys this property. No hyperbolic two-bridge knot, however, admits a lens space surgery [19].

Appendices

Appendix A

Surgery Descriptions of Berge Knots

Here we give descriptions for the families of Berge knots other than those in $\mathcal{K}_\eta$ as surgeries on the minimally twisted five chain link, MT5C $= C(5)$, or its reflection. The knots in $\mathcal{K}_\eta$ are described as surgeries on the minimally twisted chain links $\pm C(2n+1)$ in Section 3.3.2. Please refer to [3] for Berge's descriptions of these knots and the notation conventions.

A.1 Conventions.

We pass between Berge's R-R diagrams and surgery descriptions of curves on a genus 2 Heegaard surface according to the correspondence shown in Figure A.1. The other half is obtained by rotating the pictures 180° in the plane of the page.

A.2 The knots arising from knots in solid tori with surgeries yielding solid tori.

A.2.1 Type (I) Torus knots.

We pass from Berge's diagram (Figure A.2) to a corresponding realization of these knots k on Heegaard surfaces via surgeries (Figure A.3). In Figure A.3,

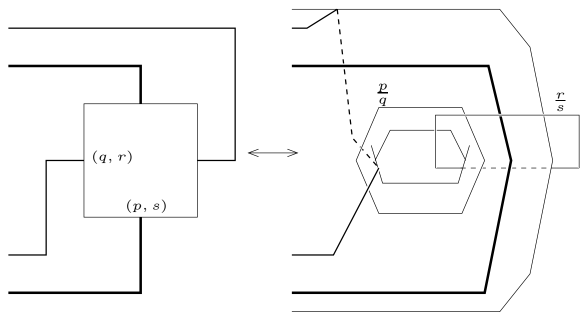


Figure A.1: Passing from a R-R diagrams to a surgery description.

$\frac{x}{y} = \frac{1}{0}$ corresponds to the meridional filling of k while $\frac{x}{y} = -\frac{1}{1}$ corresponds to the lens space surgery of k. Choose p, q, r, s so that $|ps - qr| = 1$.

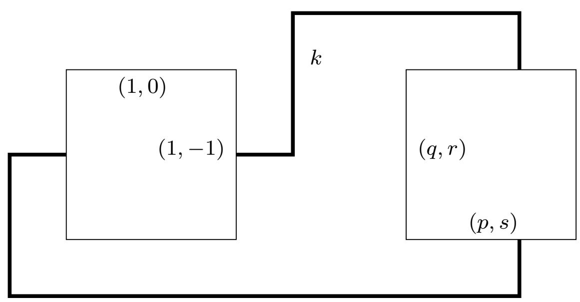


Figure A.2: Berge's R-R diagram for type (I) knots (torus knots).

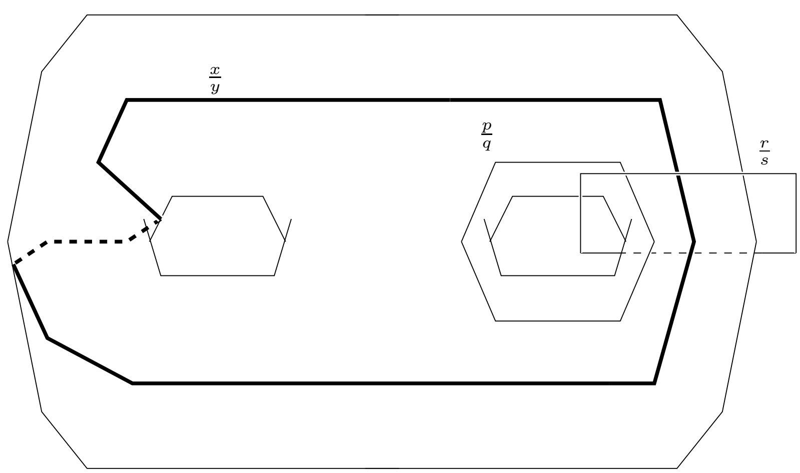


Figure A.3: Type (I) knots on Heegaard surface via surgeries.

After dropping the Heegaard surface from the picture,we add components with meridional framings, perform isotopies, and do Kirby Calculus as shown in Figure A.4 to get a description of k as surgery on the MT5C.

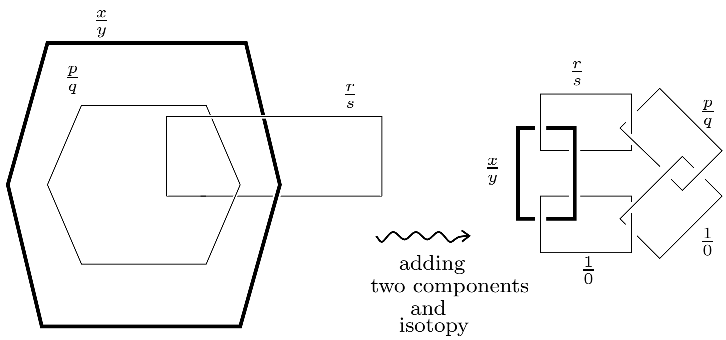


Figure A.4: Type (I) knots. Isotopies and Kirby Calculus moves ending in the MT5C.

A.2.2 Type (II) Cables about torus knots.

We pass from Berge's diagram (Figure A.5) to a corresponding realization of these knots k on Heegaard surfaces via surgeries (Figure A.6). In Figure A.6, $\frac{x}{y} = \frac{1}{0}$ corresponds to the meridional filling of k while $\frac{x}{y} = -\frac{1}{1}$ corresponds to the lens space surgery of k. Choose p, q, r, s so that $|ps - qr| = 1$.

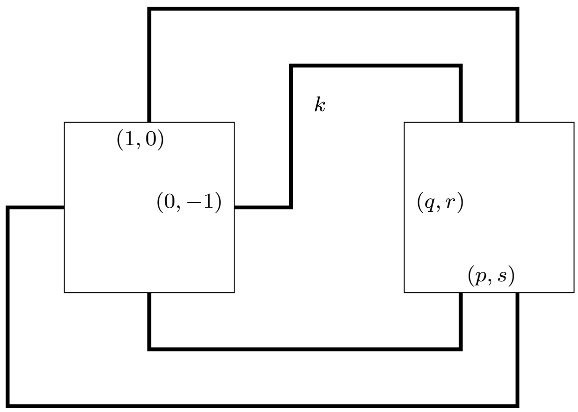


Figure A.5: Berge's R-R diagram for type (II) knots (cables about torus knots).

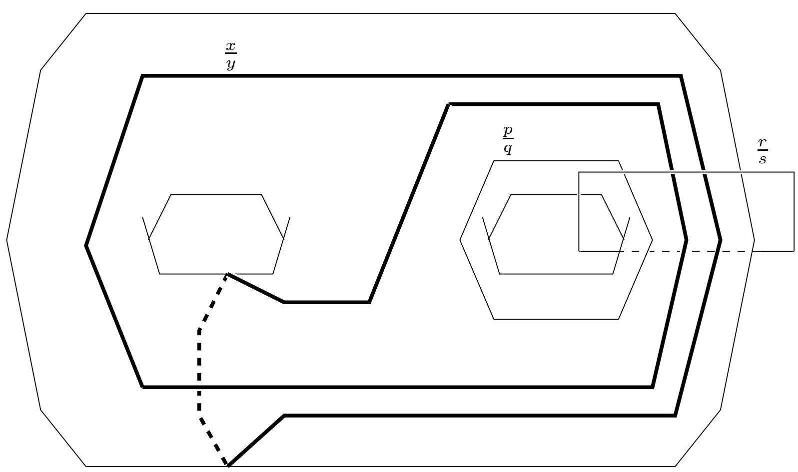


Figure A.6: Type (II) knots on Heegaard surface via surgeries.

After dropping the Heegaard surface from the picture, we add components with meridional framings, perform isotopies, and do Kirby Calculus as shown in Figure A.7 to get a description of k as surgery on the MT5C. Notice that the surgery of $+\frac{1}{1}$ on the component corresponding to k is the lens space surgery. The trivial (i.e. S^3) surgery is still $\frac{1}{0}$.

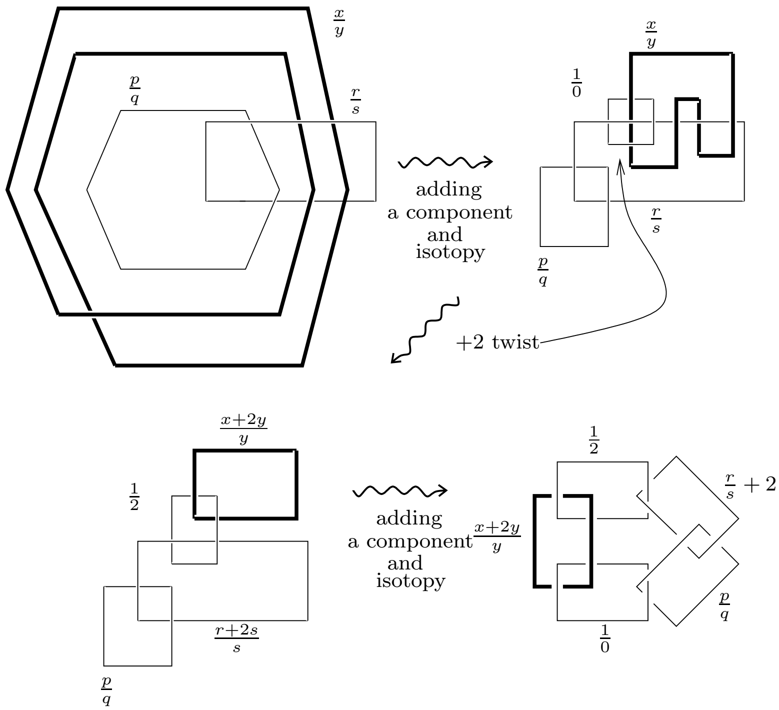


Figure A.7: Type (II) knots. Isotopies and Kirby Calculus moves ending in the MT5C.

A.2.3 Type (III).

We pass from Berge's diagram (Figure A.8) to a corresponding realization of these knots k on Heegaard surfaces via surgeries (Figure A.9). In Figure A.9, $\frac{x}{y} = \frac{1}{0}$ corresponds to the meridional filling of k while $\frac{x}{y} = -\frac{1}{1}$ corresponds to the lens space surgery of k. Choose $p \in \mathbb{N}$ and $\epsilon = \pm 1$.

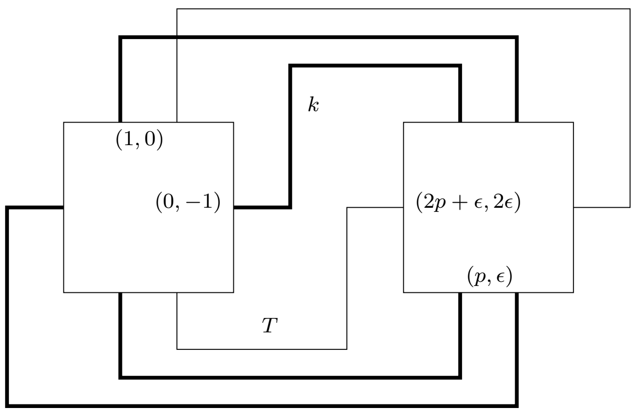


Figure A.8: Berge's R-R diagram for type (III) knots.

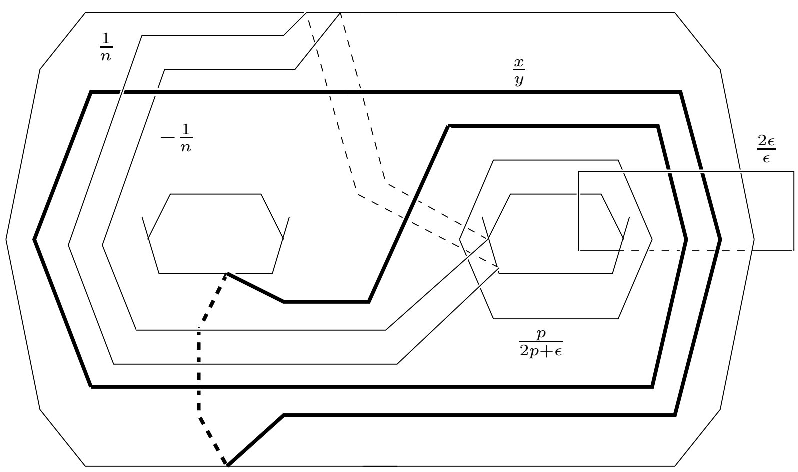


Figure A.9: Type (III) knots on Heegaard surface via surgeries.

The surgeries on the two parallel components in Figure A.10 can be amalgamated into a surgery on a single component. We then drop the Hee-

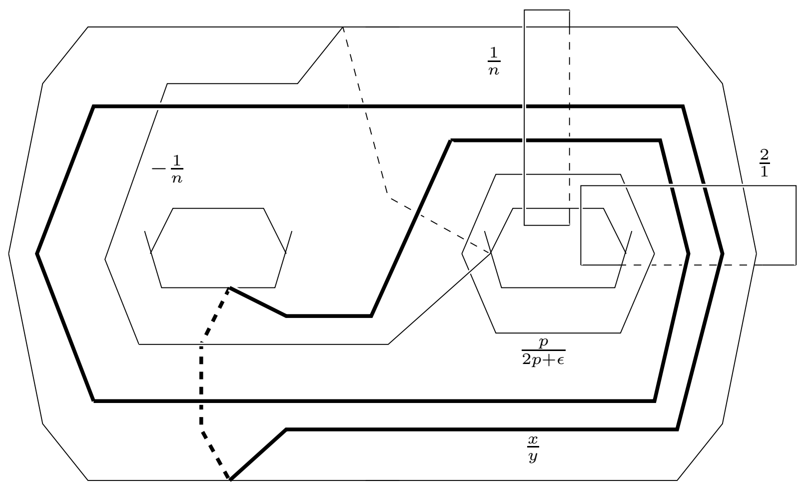


Figure A.10: Type (III) knots. Isotopy to show parallel components.

gaard surface to get the first link of Figure A.11. From here we perform isotopies and Kirby Calculus to arrive at the MT5C. Notice that the surgery of $+\frac{1}{1}$ on the component corresponding to k on the MT5C is the lens space surgery. The trivial (i.e. S^3) surgery is still $\frac{1}{0}$.

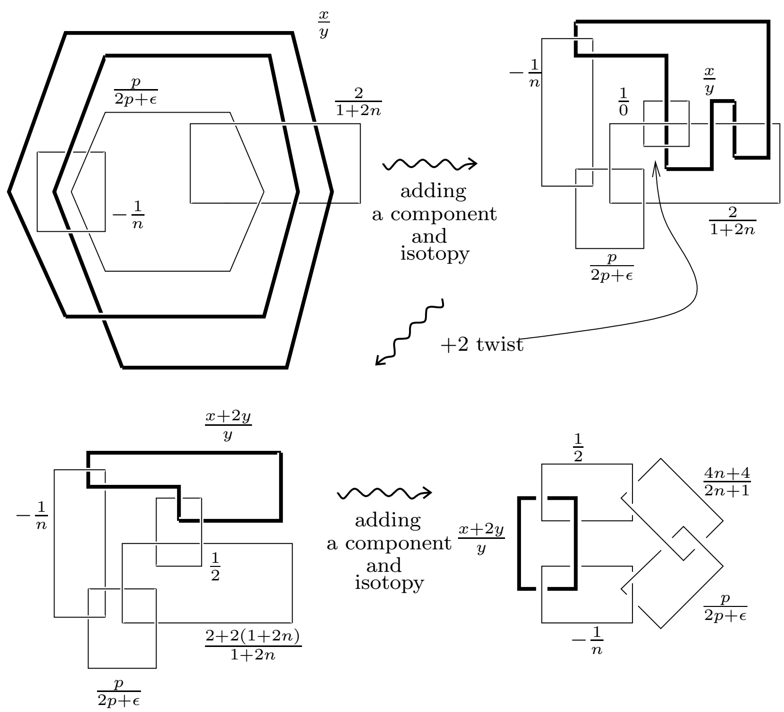


Figure A.11: Type (III) knots. Isotopies and Kirby Calculus moves ending in the MT5C.

A.2.4 Type (IV).

We pass from Berge's diagram (Figure A.12) to a corresponding realization of these knots k on Heegaard surfaces via surgeries (Figure A.13). In Figure A.13, $\frac{x}{y} = \frac{1}{0}$ corresponds to the meridional filling of k while $\frac{x}{y} = -\frac{1}{1}$ corresponds to the lens space surgery of k. Choose $p \in \mathbb{N}$ and $\epsilon = \pm 1$.

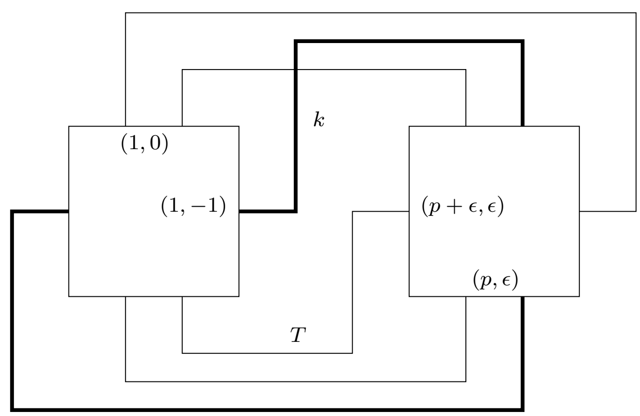


Figure A.12: Berge's R-R diagram for type (IV) knots.

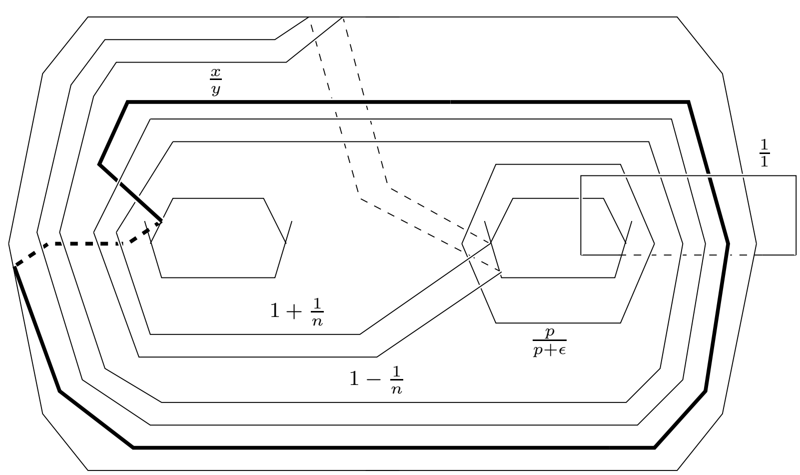


Figure A.13: Type (IV) knots on Heegaard surface via surgeries.

After dropping the Heegaard surface from the picture, we add components with meridional framings, perform isotopies, and do Kirby Calculus as shown in Figure A.14 and continued in Figure A.15 to get a description of k as surgery on the MT5C. Notice that the surgery of $\frac{0}{1}$ on the component corresponding to k is the lens space surgery. The trivial (i.e. S^3) surgery is still $\frac{1}{0}$.

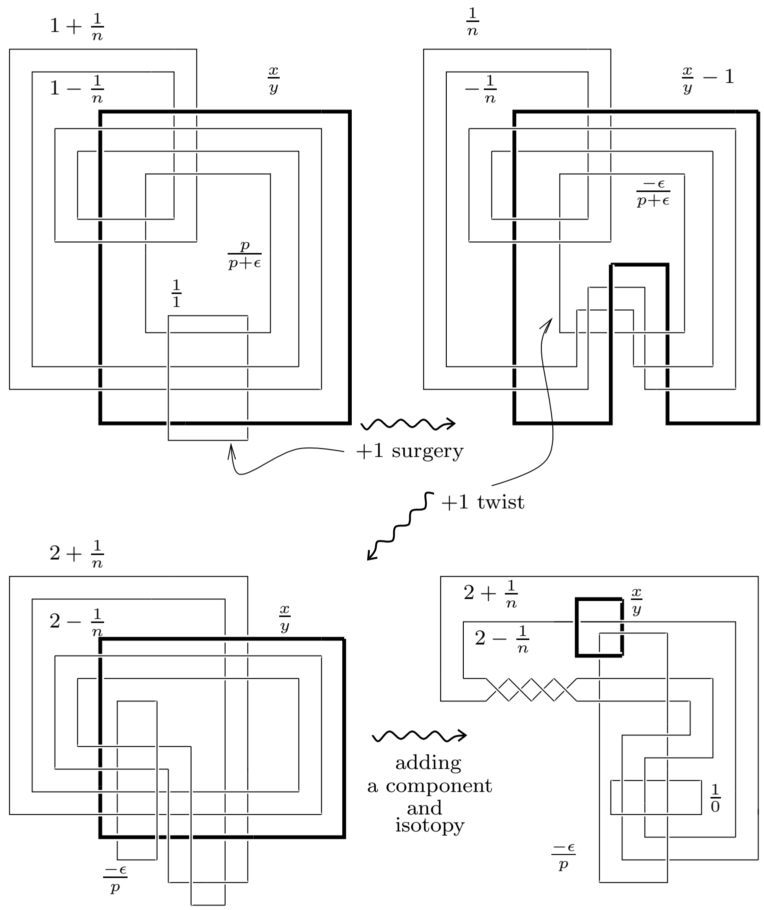


Figure A.14: Type (IV) knots. Isotopies and Kirby Calculus moves.

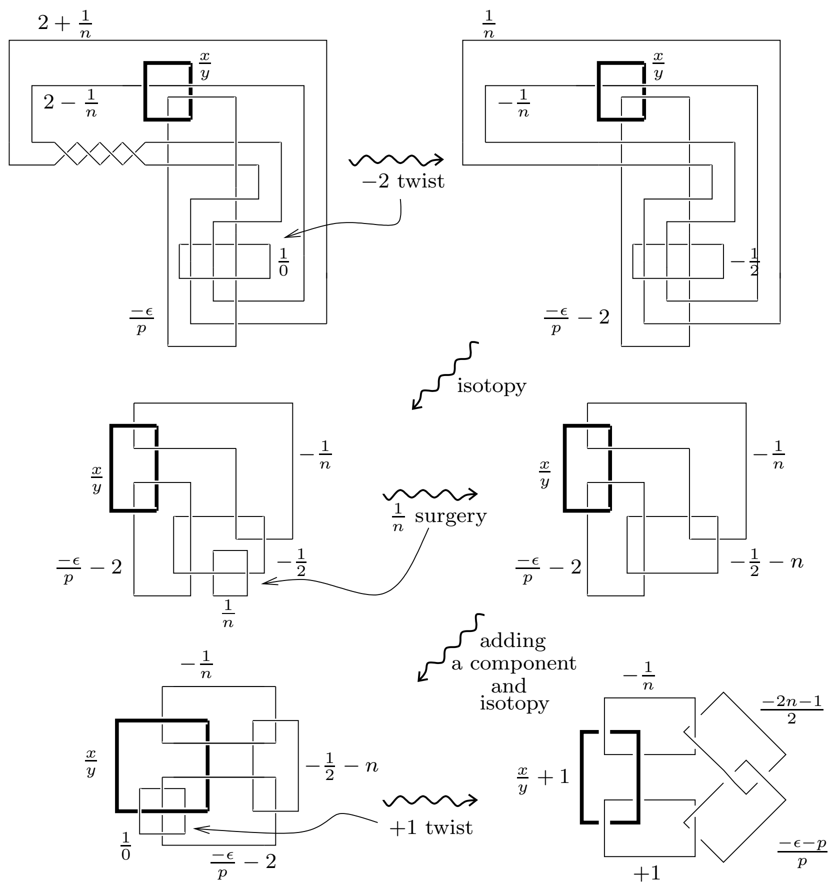


Figure A.15: Type (IV) knots. Isotopies and Kirby Calculus moves ending in the MT5C.

A.2.5 Type (V).

We pass from Berge's diagram (Figure A.16) to a corresponding realization of these knots k on Heegaard surfaces via surgeries (Figure A.17). In Figure A.17, $\frac{x}{y} = \frac{1}{0}$ corresponds to the meridional filling of k while $\frac{x}{y} = -\frac{5}{1}$ corresponds to the lens space surgery of k. Choose $p \in \mathbb{N}$ and $\epsilon = \pm 1$.

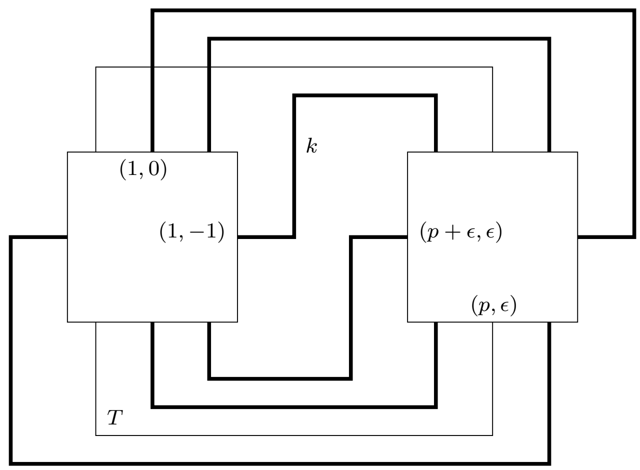


Figure A.16: Berge's R-R diagram for type (V) knots.

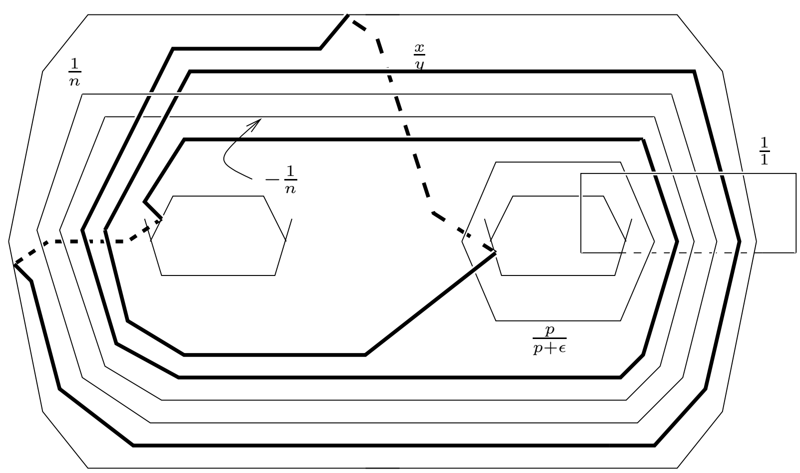


Figure A.17: Type (V) knots on Heegaard surface via surgeries.

After dropping the Heegaard surface from the picture, we add components with meridional framings, perform isotopies, and do Kirby Calculus as shown in Figure A.18 and continued in Figure A.19 to get a description of k as surgery on the MT5C. Notice that the surgery of $\frac{0}{1}$ on the component corresponding to k is the lens space surgery. The trivial (i.e. S^3) surgery is still $\frac{1}{0}$.

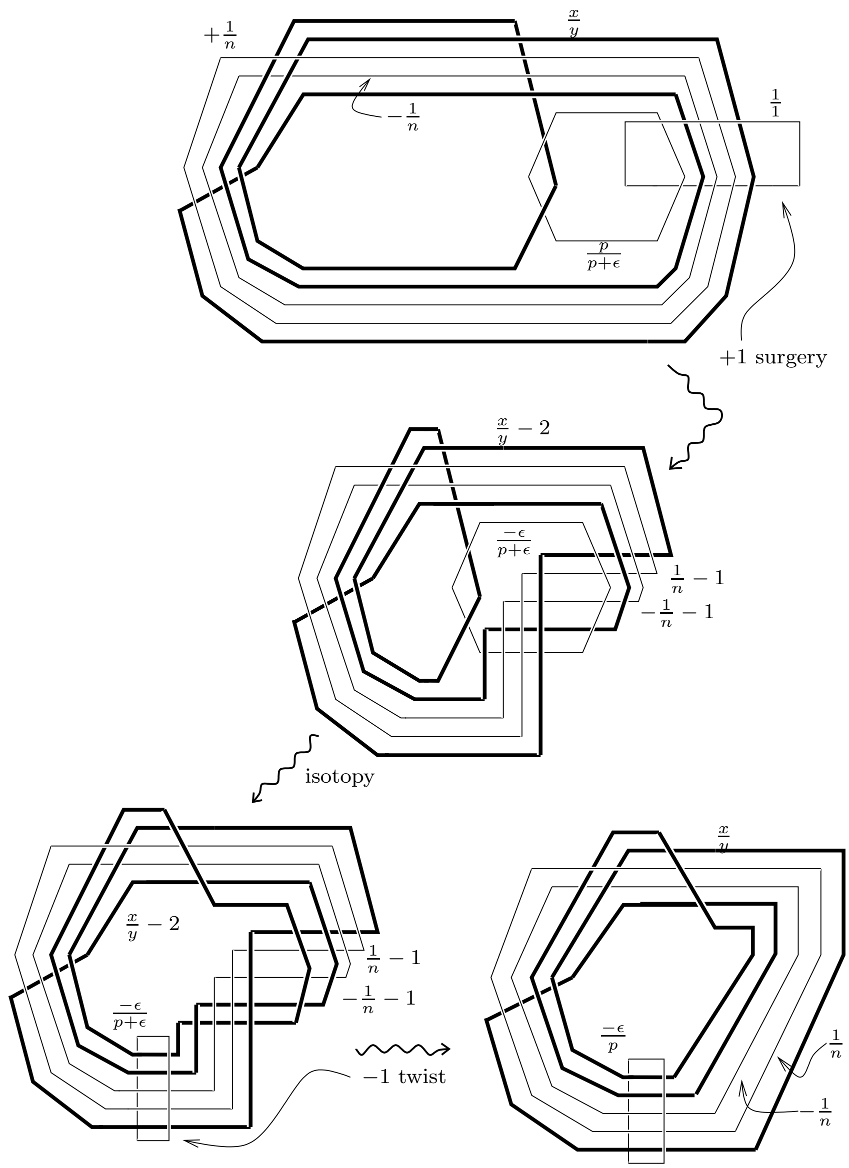


Figure A.18: Type (V) knots. Isotopies and Kirby Calculus moves.

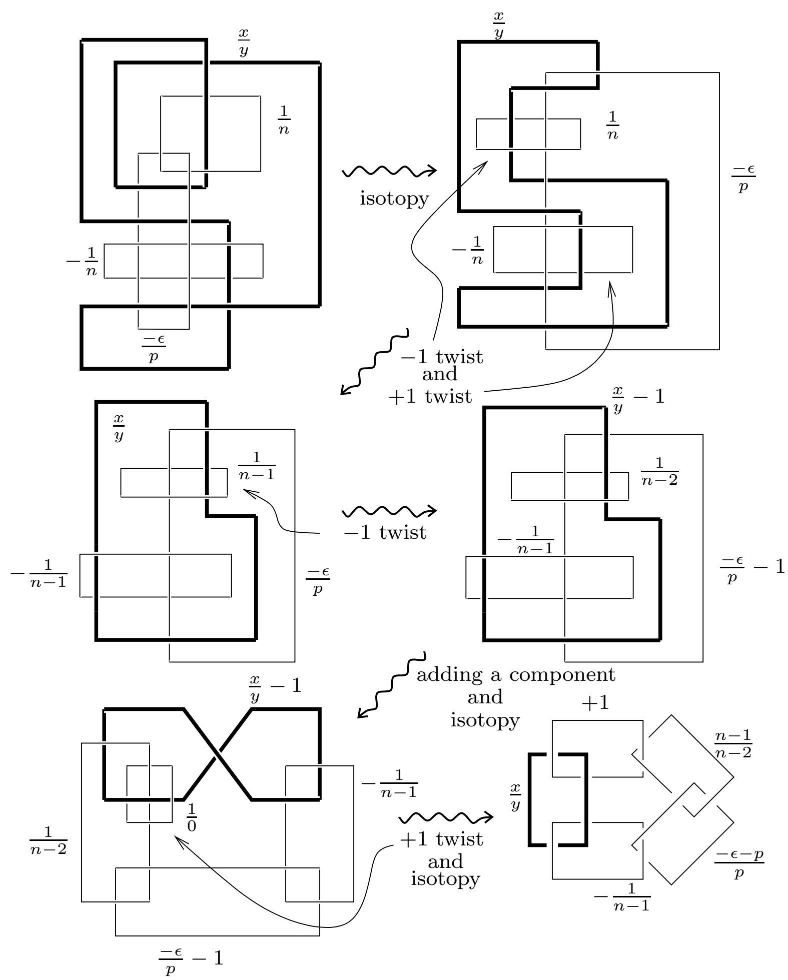


Figure A.19: Type (V) knots. Isotopies and Kirby Calculus moves ending in the MT5C.

A.2.6 Type (VI).

We pass from Berge's diagram (Figure A.20) to a corresponding realization of these knots k on Heegaard surfaces via surgeries (Figure A.21). In Figure A.21, $\frac{x}{y} = \frac{1}{0}$ corresponds to the meridional filling of k while $\frac{x}{y} = -\frac{5}{1}$ corresponds to the lens space surgery of k.

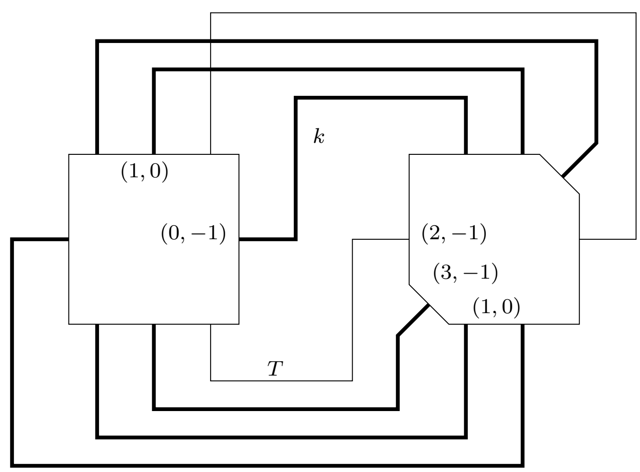


Figure A.20: Berge's R-R diagram for type (VI) knots.

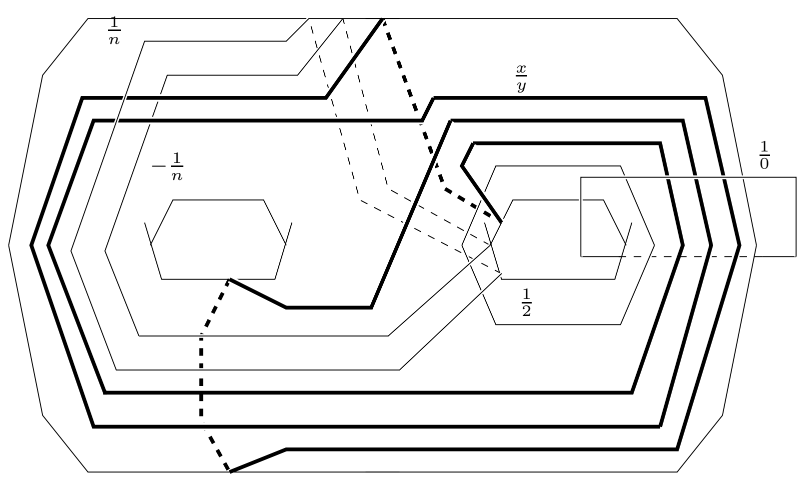


Figure A.21: Type (VI) knots on Heegaard surface via surgeries.

After dropping the Heegaard surface from the picture, we combine some of the twisting of k via the surgery of another component as shown in Figure A.22. We continue, after an isotopy, with Kirby Calculus in Figure A.23 to get a description of k as surgery on the MT5C. Notice that the surgery of $\frac{0}{1}$ on the component corresponding to k is the lens space surgery. The trivial (i.e. S^3) surgery is still $\frac{1}{0}$.

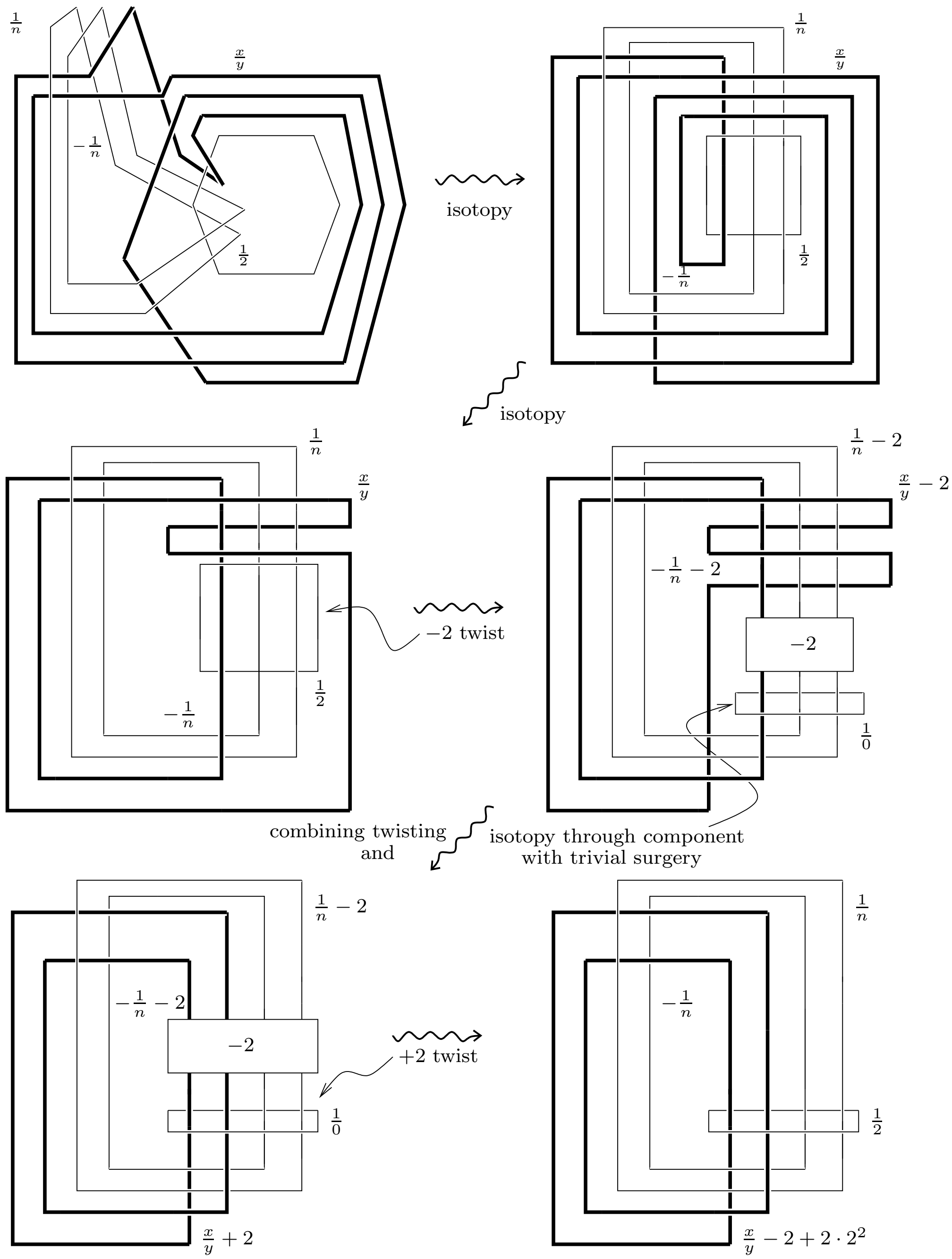


Figure A.22: Type (VI) knots. Isotopies and Kirby Calculus moves.

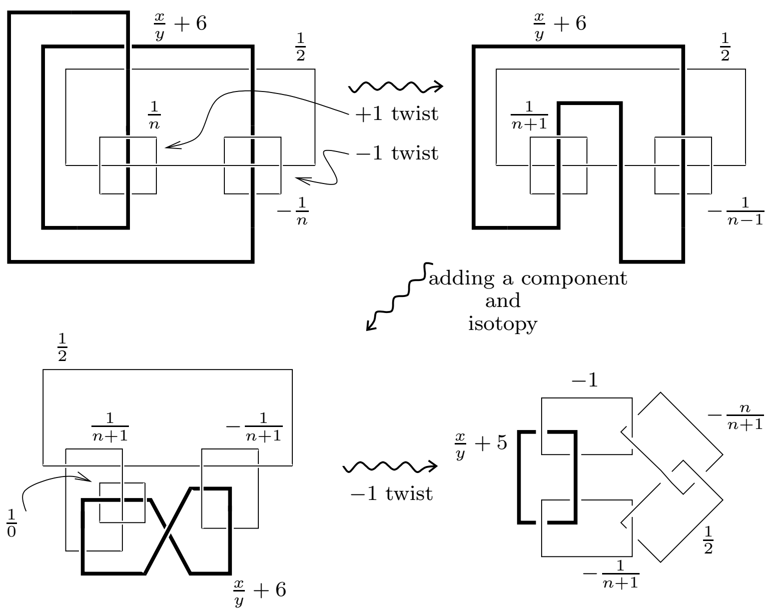


Figure A.23: Type (VI) knots. Isotopies and Kirby Calculus moves ending in the MT5C.

A.3 Sporadic knots

A.3.1 Sporadic knots type a) and b).

We pass from Berge's diagram (Figure A.24) to a corresponding realization of these knots k on Heegaard surfaces via surgeries (Figure A.25). In Figure A.25, $\frac{x}{y} = \frac{1}{0}$ corresponds to the meridional filling of k while $\frac{x}{y} = -\frac{1}{1}$ corresponds to the lens space surgery of k.

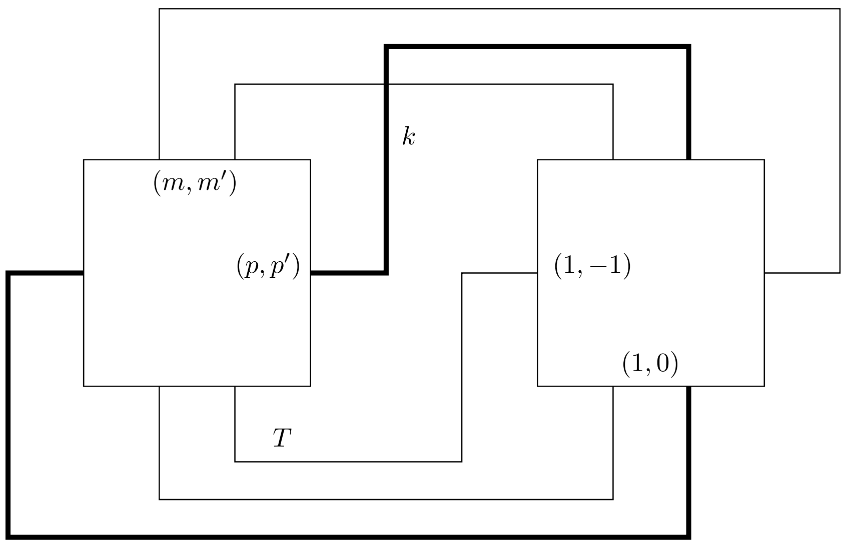


Figure A.24: Berge's sporadic knot types a) and b) diagram.

As shown in Figure A.26, one component bounds a Möbius band. The framing on that component induced by the Möbius band agrees with the framing from the surface.

In Figure A.27, we replace the component that bounds the Möbius band with the core curve of the Möbius band and the corresponding surgery instructions. We continue, after an isotopy, with Kirby Calculus in Figure A.28 to get a description of k as surgery on the MT5C.

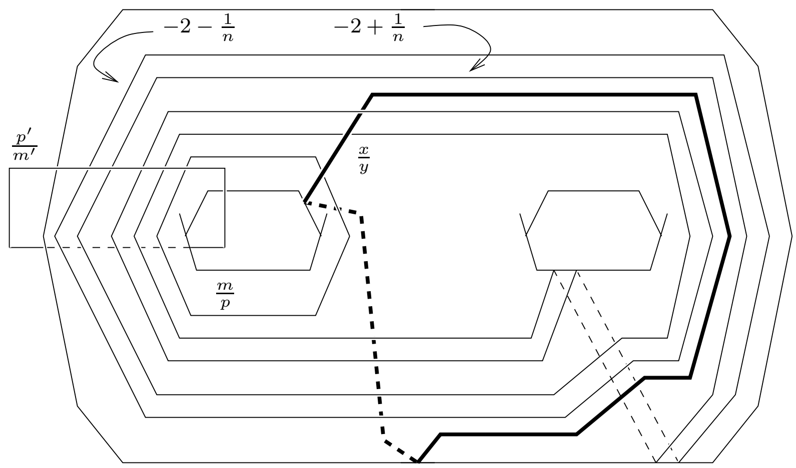


Figure A.25: Sporadic knot types a) and b) on Heegaard surface via surgeries.

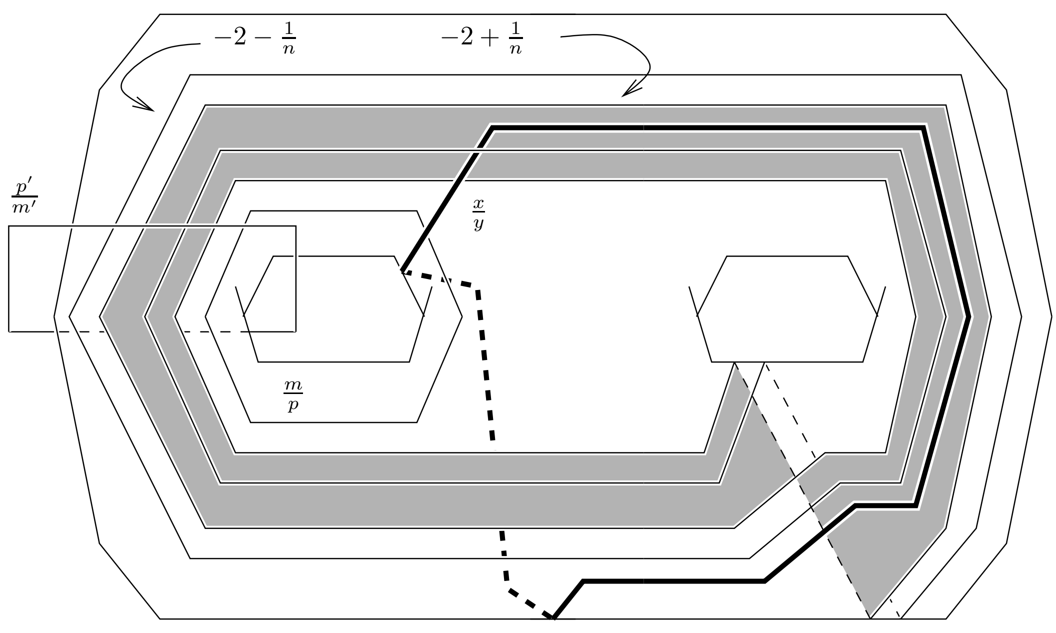


Figure A.26: Sporadic types a) and b). One component bounds a Möbius band.

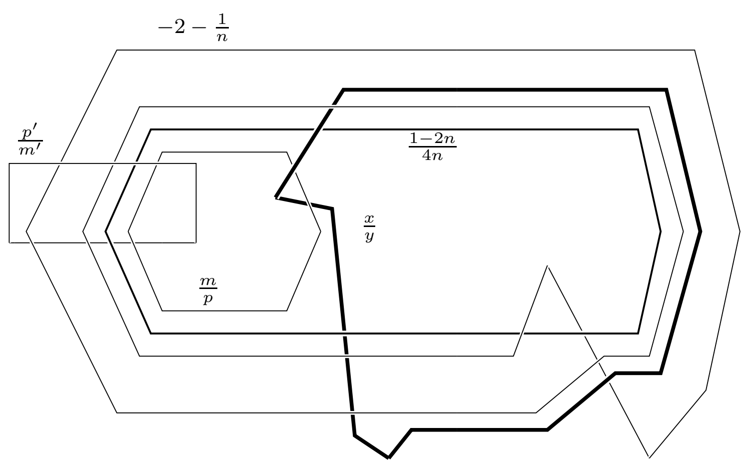


Figure A.27: Sporadic types a) and b). Core curve of Möbius band replaces the boundary of Möbius band.

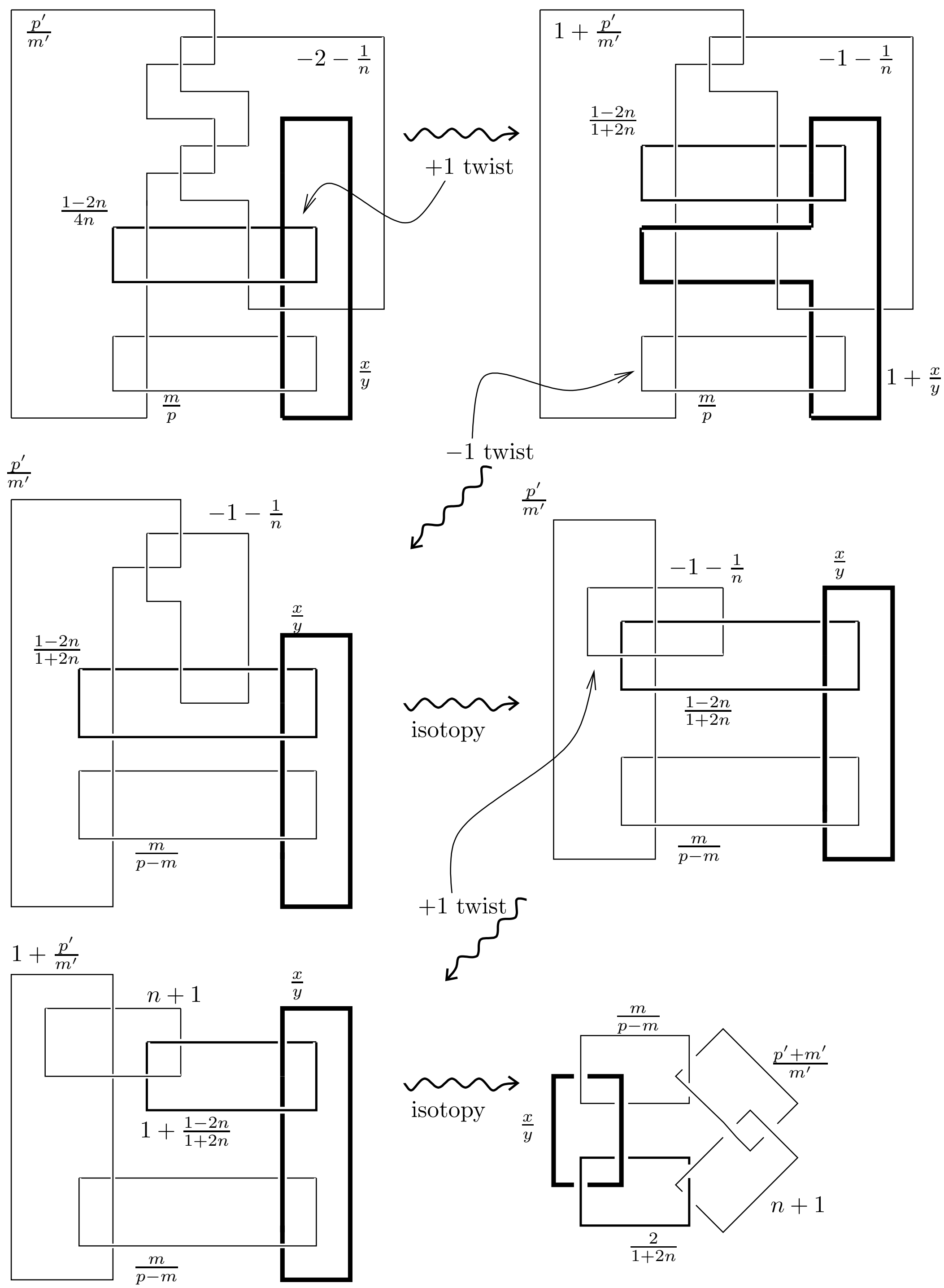


Figure A.28: Sporadic types a) and b). Isotopies and Kirby Calculus moves ending in the MT5C.

A.3.2 Sporadic knots type c) and d).

We pass from Berge's diagram (Figure A.29) to a corresponding realization of these knots k on Heegaard surfaces via surgeries (Figure A.30). In Figure A.30, $\frac{x}{y} = \frac{1}{0}$ corresponds to the meridional filling of k while $\frac{x}{y} = \frac{0}{1}$ corresponds to the lens space surgery of k.

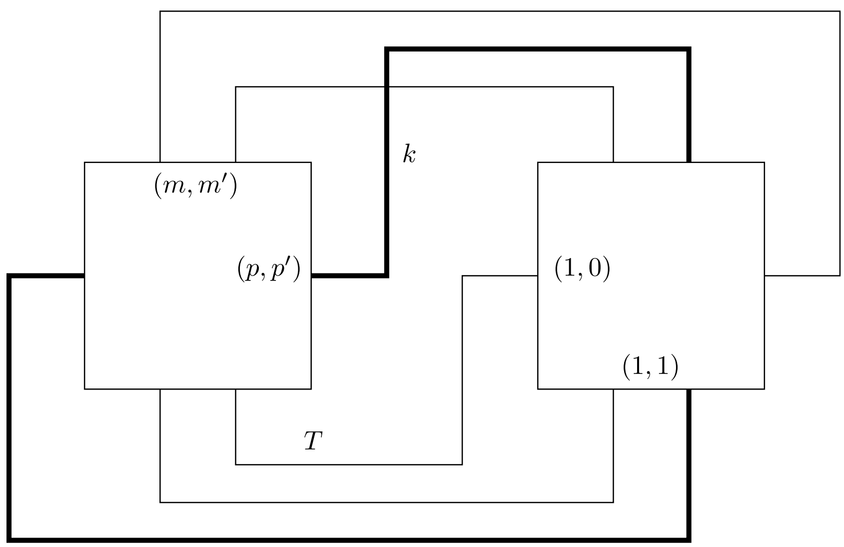


Figure A.29: Berge's sporadic knot types c) and d) diagram.

As shown in Figure A.31, one component bounds a Möbius band. The framing on that component induced by the Möbius band agrees with the framing from the surface.

In Figure A.32, we replace the component that bounds the Möbius band with the core curve of the Möbius band and the corresponding surgery instructions. We continue, after an isotopy, with Kirby Calculus in Figure A.33 to get a description of k as surgery on the MT5C.

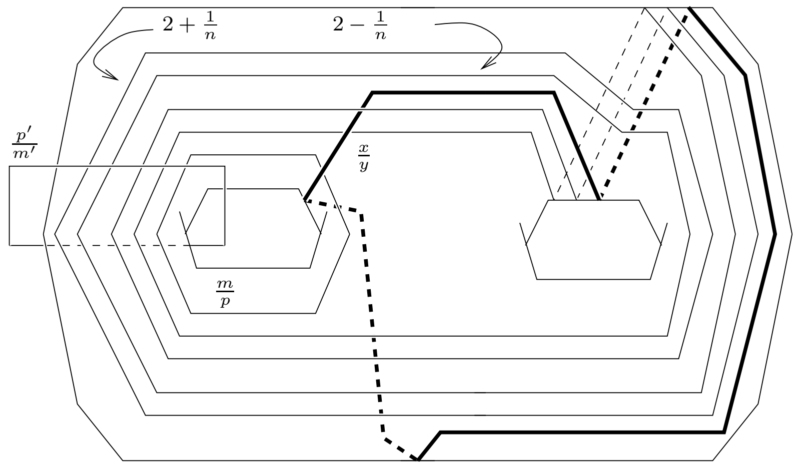


Figure A.30: Sporadic knot types c) and d) on Heegaard surface via surgeries.

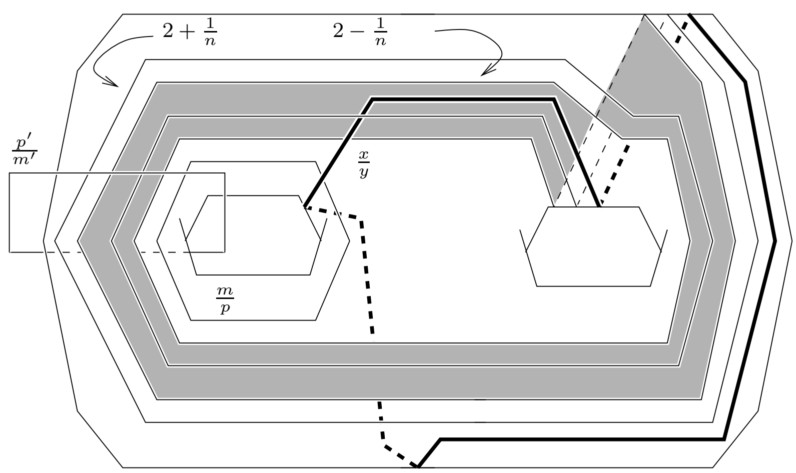


Figure A.31: Sporadic types c) and d). One component bounds a Möbius band.

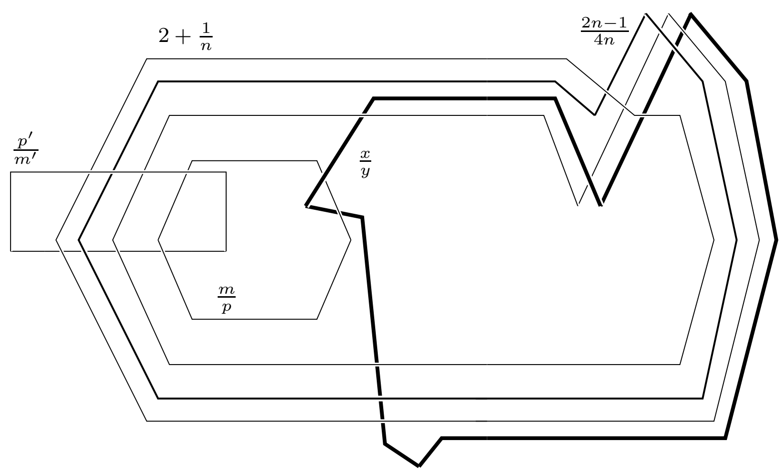


Figure A.32: Sporadic types c) and d). Core curve of Möbius band replaces the boundary of Möbius band.

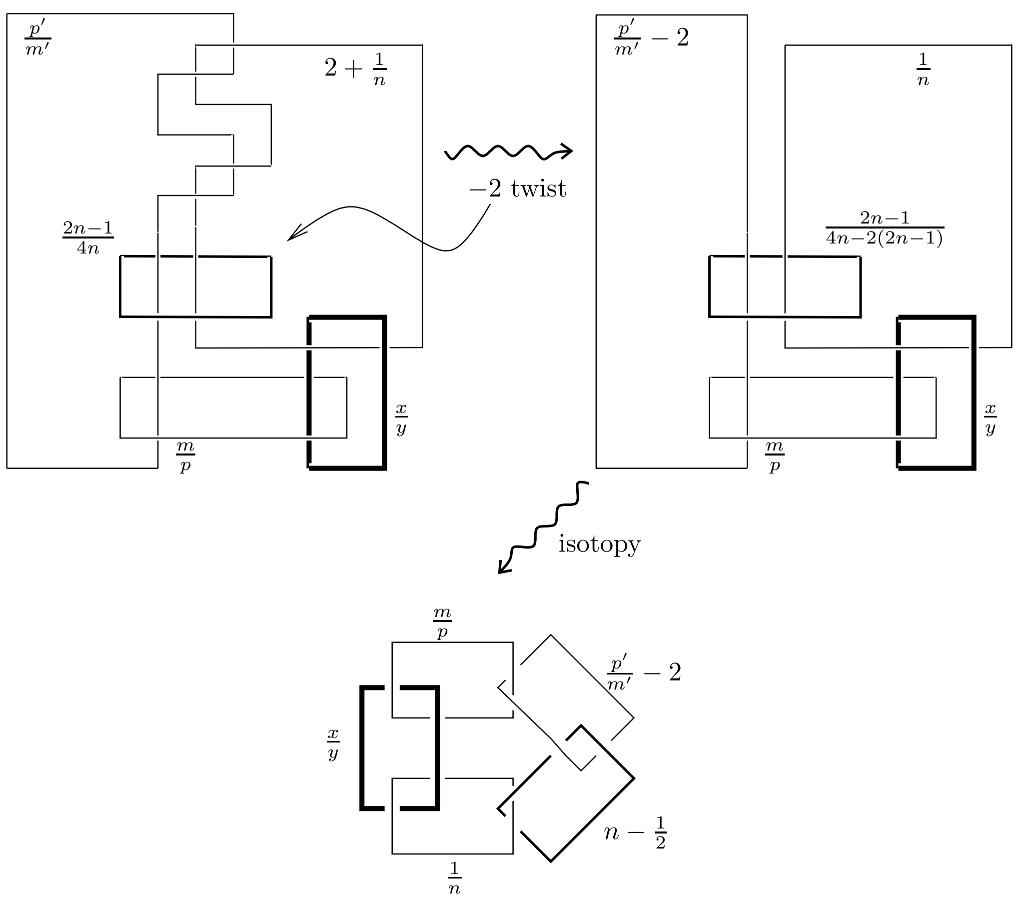


Figure A.33: Sporadic types c) and d). Isotopies and Kirby Calculus moves ending in the MT5C.

A.4 Summary of Berge knots as surgeries on the MT5C.

Figure A.34 shows Berge knot types I) - VI) as surgeries on the MT5C. Figure A.35 shows Berge sporadic knot types a) - d) as surgeries on the MT5C. The component corresponding to the Berge knot is shown with a pair of surgeries $(\rho_{S^3}, \rho_{\mathrm{LensSp}})$ where ρ_{S^3} is the surgery slope yielding S^3 and ρ_{LensSp} is the surgery slope yielding the lens space.

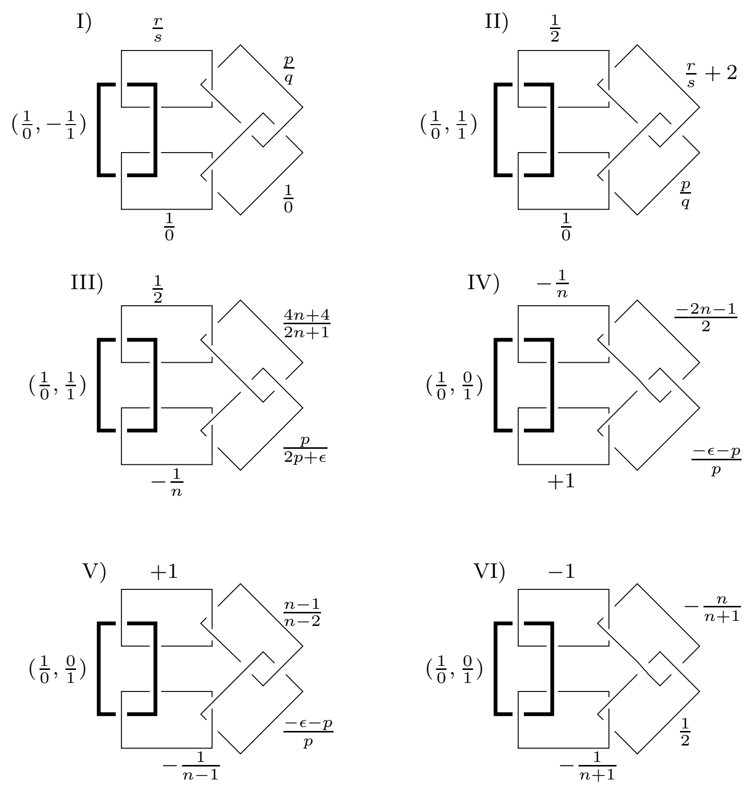


Figure A.34: Berge knots Types I-VI

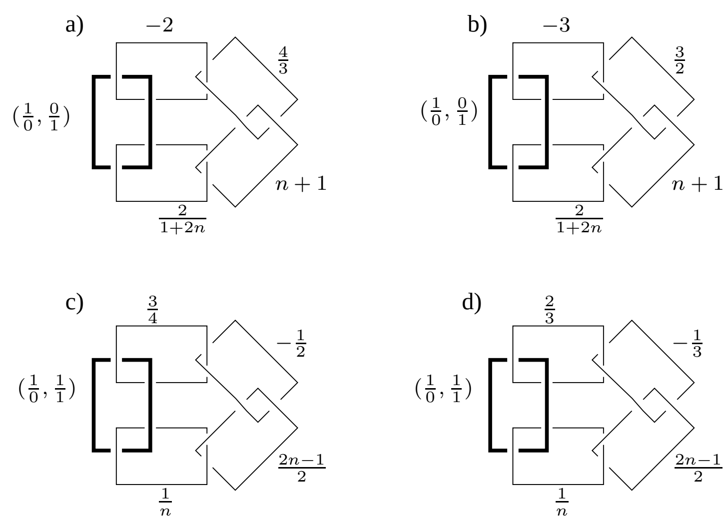


Figure A.35: Sporadic Berge Knots

Appendix B

Positive Braids and Knots in $\mathcal{K}_\eta$

We show an the following interesting fact about the structure of the Berge knots $\mathcal{K}_\eta \subset S^3$, the knots that lie on the fiber of a trefoil or the Figure Eight Knot.

Theorem B.0.1. *The knots that lie on the fiber of a genus one fibered knot may be expressed as the closure of a positive or negative braid.*

In this appendix we assume the reader's familiarity with braids and train tracks. We refer the reader to one of the many books on Knot Theory (e.g. [5], [11]) for an overview of braids and to [16] for an overview of train tracks. Throughout this appendix we will use the language of braids in the context of train tracks with surfaces.

B.1 Proof of Theorem B.0.1

Proof. The genus one fibered knots are the Left Handed Trefoil, the Right Handed Trefoil, and the Figure Eight Knot [4]. We will prove this theorem explicitly for the Left Handed Trefoil and the Figure Eight Knot as the case of the Right Handed Trefoil is obtained from that of the Left Handed Trefoil by reflection.

The once-punctured torus fiber surfaces T of the Left Handed Trefoil and Figure Eight Knot may be viewed as in Figure B.1 (as the plumbing of two Hopf bands). Boxes labeled $+1$ (resp. -1) indicate a positive (resp. negative) twist. See Figure B.2.

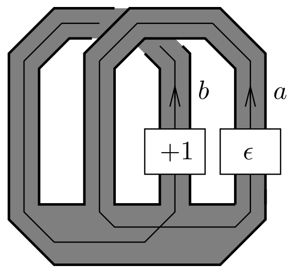


Figure B.1: Plumbing of Hopf bands

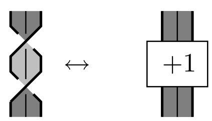


Figure B.2: Full positive twist

The Left Handed Trefoil corresponds to choosing $\epsilon = +1$ in Figure B.1, and the Figure Eight Knot corresponds to choosing $\epsilon = -1$. Note the oriented simple closed curves a and b on T that form a basis for $H_1(T;\mathbb{Z})$.

Let c be an essential simple closed curve on T with slope $\frac{p}{q}$. (Thus for some orientation on c, $[c] = p[a] + q[b] \in H_1(T)$.) Then c is carried by one of the four train tracks τ_i, $i \in \{1, 2, 3, 4\}$, of Figure B.3 depending on the slope of c. We indicate the weights on the train tracks corresponding to the curve c it carries. The third arc may be left unlabeled since its weight is implied by the switching conditions.

We may redraw a portion of the surface T with τ_i in braid like form

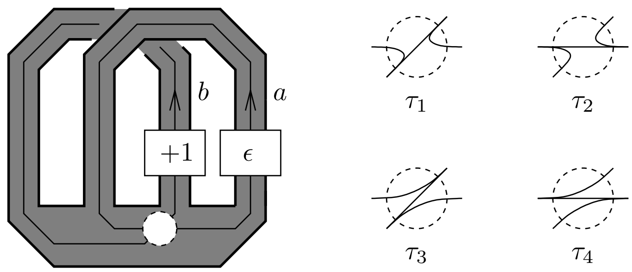


Figure B.3: The four train tracks.

so that the corresponding "closure" is the surface T with τ_i. See Figure B.4. Furthermore, curves carried by τ_i may then be seen as the closure of a braid.

Notice that Figures B.4 (a) and (b) are splittings of Figure B.5 (a) which is "conjugate" to Figure B.5 (b). Similarly, Figures B.4 (c) and (d) are splittings of Figure B.6 (a) which is "conjugate" to Figure B.6 (b).

We will use the moves of Figure B.7. The second move is akin to the Markov move for braids and their closures. Also note that if the surface and train track are positioned as the form of the closure of a positive (or negative) braid, then any curve carried by the train track will be the closure of a positive (or negative) braid.

Case 1. The Left Handed Trefoil, $\epsilon = +1$.

If c is carried by train tracks τ_1 or τ_2, then it is carried by the train track in Figure B.5 (b). Since $\epsilon = +1$, this train track has the form of a positive braid. Thus c is the closure of a positive braid.

If c is carried by train tracks τ_3 or τ_4, then notice that since $\epsilon = +1$, there is a flip symmetry between Figures B.4 (c) and (d). Therefore we only need show that if c is carried by τ_3 then it is the closure of a positive braid.

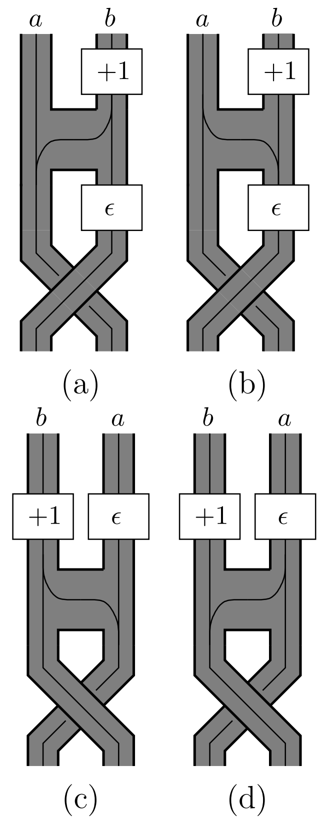


Figure B.4: The four train tracks in braid position.

Consider the sequence of moves in Figure B.8 beginning with Figure B.4 (c). The resulting train track from this sequence is the train track of Figure B.5 (b) with different weights. Thus the curve c is the closure of a positive braid.

It thus follows that all the essential simple closed curves on the fiber of the Left Handed Trefoil are closures of positive braids.

By reflection through the plane of the page, we similarly see that all the essential simple closed curves on the fiber of the Right Handed Trefoil are closures of negative braids.

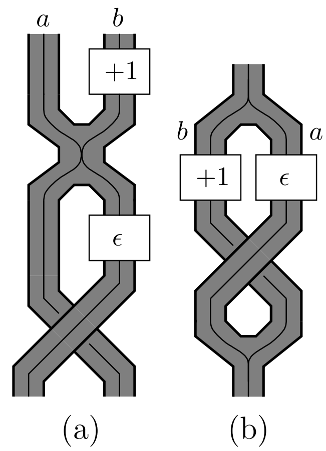


Figure B.5: Pre-split train tracks.

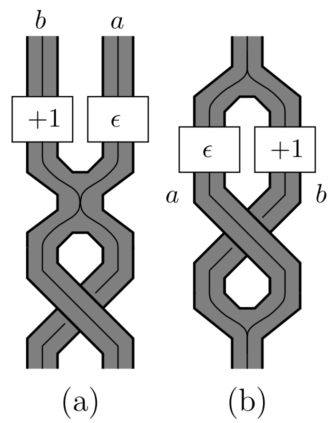


Figure B.6: Pre-split train tracks

Case 2. The Figure Eight Knot, $\epsilon = -1$.

First observe that by reflecting through the plane of the page, Figure B.4 (d) may be obtained from Figure B.4 (a) and Figure B.4 (c) may be obtained from Figure B.4 (b). Therefore we will direct our attentions to Figures B.4 (a) and (b) and the curves carried by τ_1 or τ_2.

Subcase 2.a Let c be carried by τ_1. Consider the sequence of moves in Figure B.9 beginning with Figure B.4 (a).

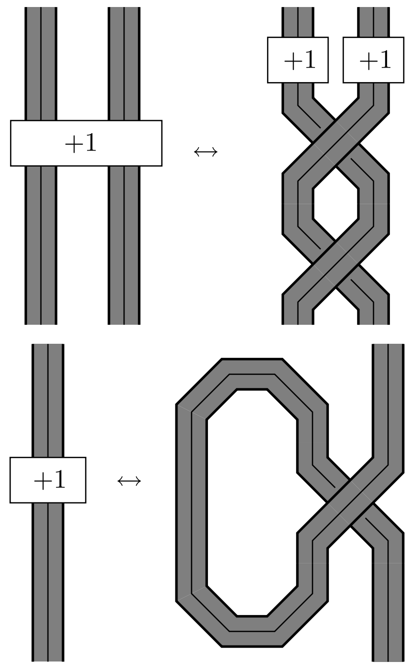


Figure B.7: Moves for braiding train tracks on surfaces.

This move sequence concludes with the the train track being in the form of the closure of a positive braid. Thus c is the closure of a positive braid. (If c was carried by τ_4, then due to the reflection, it is the closure of a negative braid.)

Subcase 2.b Let c be carried by τ_2. Consider the sequence of moves in Figure B.10 beginning with Figure B.4 (b). Notice the effect on the weights.

Flipping the result over we get Figure B.6 (b) but with smaller total weight. After a reflection of this through the plane of the page we have Figure B.5 (b) but with smaller total weight. Thus c equivalent (up to reflection) to a curve carried by τ_1 or τ_2 but with smaller total weight.

Since the above sequence of moves reduces the total weight on the train track, repeating this process will eventually end up in Subcase 2.a or with a curve carried by τ_2 with weight 1 on the two labeled arcs. In the latter, the curve is the unknot.

Therefore all the essential simple closed curves on the fiber of the Figure Eight Knot are closures of positive or negative braids. □

To summarize the conclusion of this theorem, any knot that may be written as an essential simple closed curve on the fiber of the Left Handed Trefoil, the Right Handed Trefoil, or the Figure Eight Knot is carried by one of the train tracks in Figure B.11. From these one can determine the corresponding positive or negative braid.

B.2 Consequences

Here we mention a few consequences of Theorem B.0.1. We also note that these knots belong to the larger family of "quasipositive" knots in which Lee Rudolph among others have had much interest.

Mind Remark 2.4.1 when correlating the following with Berge's computations of lens spaces.

B.2.1 Hopf plumbings.

By writing these knots as the closures of positive (or negative) braids, it follows that the these knots are fibered. Furthermore these knots and their fiber can be obtained by plumbing a collection of positive (or a collection of negative) Hopf bands together. See [18] for one description of how this may be done.

B.2.2 Genera.

The fiber for these knots can be obtained from applying Seifert's algorithm to the diagram of the associated closure of a positive (or negative) braid representing the knot. This makes determining the genus of the knot (i.e. the genus of the fiber) direct.

Note that a full twist on a collection of n adjacent strands of a braid produces $n(n-1)$ crossings. Furthermore, the passing of n adjacent strands over m strands produces nm crossings.

Consider a knot k carried by the closure of the braid of Figure B.4 (a) with $\epsilon = +1$. Thus k is a knot on the Left Handed Trefoil. Since this knot is already represented as a positive braid, an application of Seifert's algorithm constructs the fiber surface from $a(a-1) + b(b-1) + ab$ crossings and $a + b$ disks. Therefore the Euler characteristic of the corresponding knot k is

$$\begin{aligned}\chi(k) &= (a+b) - (a(a-1) + b(b-1) + ab) \\ &= -a^2 - ab - b^2 + 2(a+b),\end{aligned}$$

and its genus is

$$g(k) = \frac{1}{2}(1 + a^2 + ab + b^2) - (a+b).$$

A similar computation can be done for the knots carried by the closure of one of the train tracks in Figure B.11.

B.2.3 Framings.

The fiber for these knots gives the knot's standard framing, i.e. its standard longitude. By carrying the fiber of the Left Handed Trefoil and Figure Eight

Knot along with the transformations of the train track in the proof of Theorem B.0.1, we can deduce the framing induced by this surface. This framing is the Dehn surgery slope that yields a lens space [3].

Again consider a knot k carried by the closure of the braid of Figure B.4 (a) with $\epsilon = +1$. By rewriting the $+1$ twists as loops as in the second move of B.7 (i.e. "flattening out the twist"), the blackboard framing then agrees with the framing induced by the surface of the genus one fibered knot. Since all crossings are positive, we may calculate the framing by simply counting the number crossings. Note that a twist around n strands flattens out to give n^2 crossings. Therefore the blackboard framing and hence the slope of the lens space surgery for the knot k is

$$\rho_{\mathrm{LensSp}} = a^2 + ab + b^2.$$

A similar computation of surgery slope can be done for the knots carried by the closure of one of the train tracks in Figure B.11.

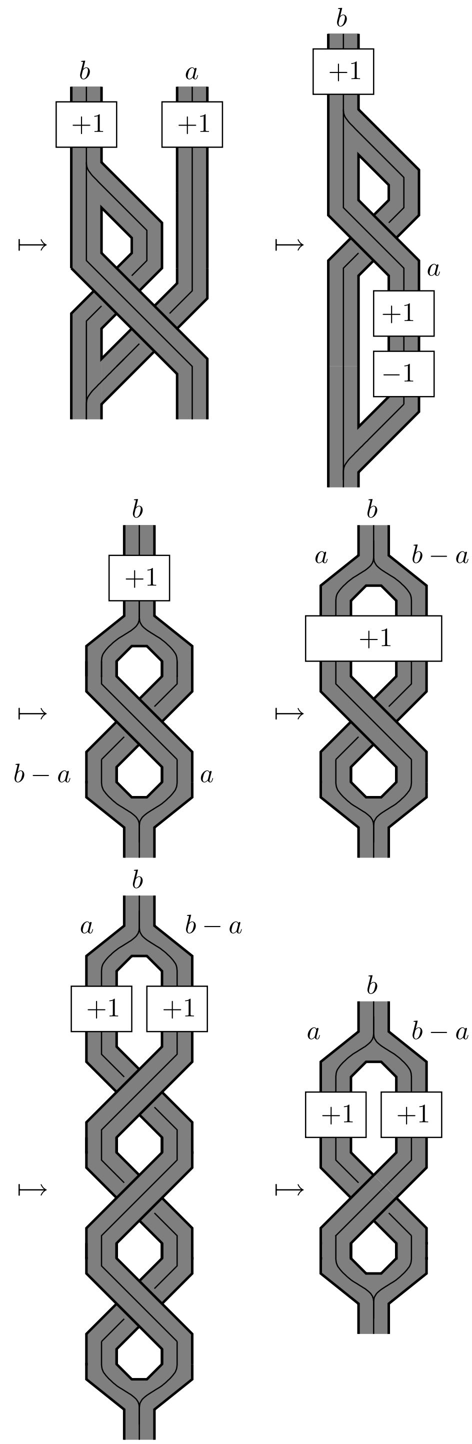


Figure B.8: Sequence of moves for τ_3 with the Left Handed Trefoil.

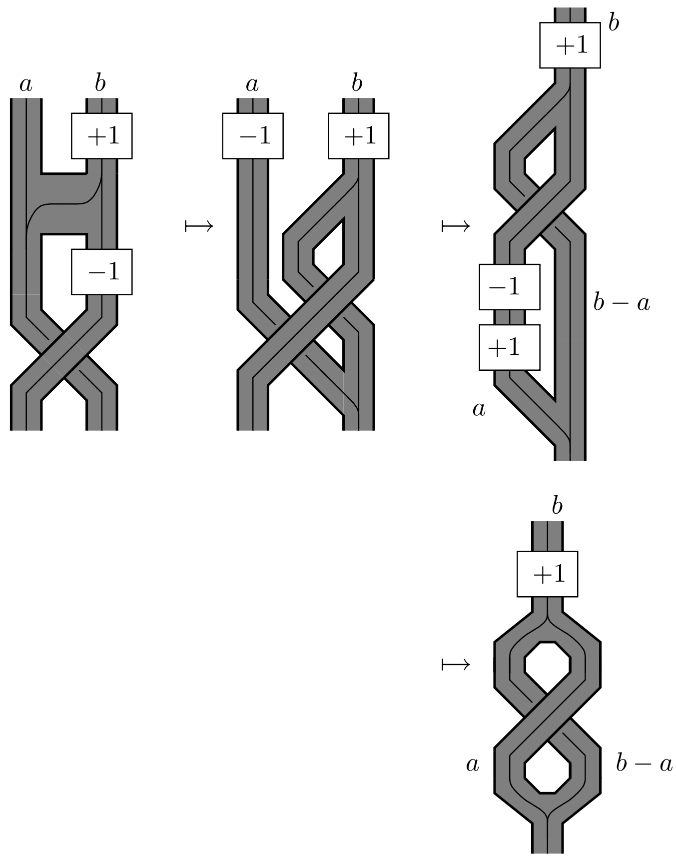


Figure B.9: Sequence of moves for τ_1 with the Figure Eight Knot.

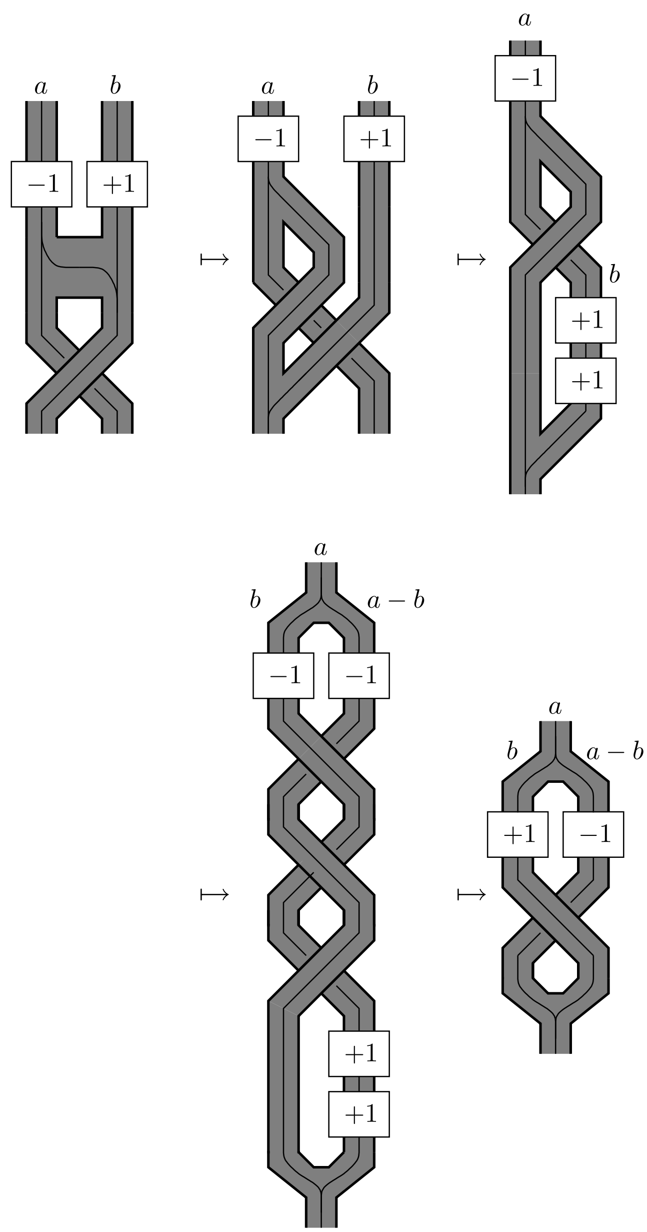
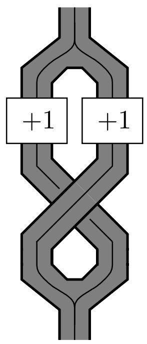
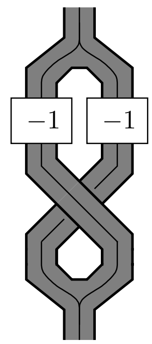


Figure B.10: Sequence of moves for τ_2 with the Figure Eight Knot.



Left Handed Trefoil (positive braid)



Right Handed Trefoil (negative braid)

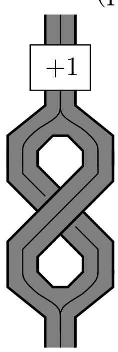


Figure Eight Knot (positive braid)

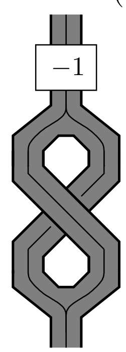


Figure Eight Knot (negative braid)

Figure B.11: Summary of the braided train tracks.

Bibliography

[1] C. Adams. Volumes of n-cusped hyperbolic 3-manifolds. *J. London Math. Soc. (2)*, 38(2):555–565, 1988.

[2] R. Benedetti and C. Petronio. *Lectures on hyperbolic geometry.* Springer-Verlag, Berlin, 1992.

[3] John Berge. Some knots with surgeries yielding lens spaces. *Alg. & Geom. Top.*, X(20XX) 1-XXX, 20XX.

[4] G. Burde and H. Zieschang. Neuwirthsche knoten und flächenabbildungen. *Abh. Math. Sem. Univ. Hamburg*, 1967.

[5] G. Burde and H. Zieschang. *Knots.* Number 5 in de Gruyter Studies in Mathematics. Walter de Gruyter & Co., 2 edition, 2003.

[6] M. Culler, W. Jaco, and H. Rubinstein. Incompressible surfaces in once-punctured torus bundles. *Proc. London Math. Soc. (3)*, 45(3):385–419, 1982.

[7] W. Floyd and A. Hatcher. Incompressible surfaces in punctured-torus bundles. *Topology Appl.*, 13(3):263–282, 1982.

[8] C. McA. Gordon and J. Luecke. Knots are determined by their complements. *Bull. Amer. Math. Soc. (N.S.)*, 1989.

[9] W. Haken. Some results on surfaces in 3-manifolds. In *Studies in Modern Topology*, pages 39–98. Math. Assoc. Amer., 1968.

[10] A. Hatcher and W. Thurston. Incompressible surfaces in 2-bridge knot complements. *Invent. Math.*, 79(2):225–246, 1985.

[11] A. Kawauchi. *A survey of knot theory.* Birkhuser Verlag, 1996.

[12] R. Kirby and P. Melvin. Dedekind sums, μ-invariants and the signature cocycle. *Math. Ann.*, 2(299):231–267, 1994.

[13] P. Kronheimer, M. Tomasz, P. Ozsváth, and Z. Szabó. Monopoles and lens space surgeries. *ArXive:math.GT/0310164*, 2003.

[14] W. B. R. Lickorish. A representation of orientable combinatorial 3-manifolds. *Ann. of Math.*, 1962.

[15] W. Neumann and A. Reid. *Topology '90*, chapter Arithmetic of hyperbolic manifolds, pages 273–310. Ohio State Univ. Math. Res. Inst. Publ., de Gruyter, Berlin, 1990.

[16] R. Penner and J. Harer. *Combinatorics of train tracks.* Number 125 in Annals of Mathematics Studies. Princeton University Press, 1992.

[17] D. Rolfsen. *Knots and Links.* Publish or Perish, Inc., 1976, 1990.

[18] L. Rudolph. Hopf plumbing, arborescent seifert surfaces, baskets, espaliers, and homogeneous braids. *Topology Appl.*, 116(3):255–277, 2001.

[19] M. Takahashi. Two-bridge knots have property p. *Mem. Amer. Math. Soc.*, 1981.

[20] W. Thurston. *The Geometry and Topology of* 3*-manifolds.* Princeton University, 1978-80.

[21] A. Wallace. Modifications and cobounding manifolds. *Canad. J. Math.*, 1960.

[22] Y. Yamada. Berge's knots in the fiber surfaces of genus one, lens spaces and framed links. Preprint, 2003.

Vita

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